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A New Class Of s-TYPE $X(u, v, l_p(E))$ Operators

Pınar Zengin Alp*1, Merve İlkhan2

Abstract

In this paper, we define a new class of s-type $X(u, v, l_p(E))$ operators, $L_{u,v,E}$. Also we show that this class is a quasi-Banach operator ideal and we study on the properties of the classes which are produced via different types of s-numbers.

Keywords: operator ideals, s-numbers, block sequence spaces.

1. INTRODUCTION

Operator ideal theory is an important subject of functional analysis. There are many different ways of constructing operator ideals, one of them is using s-numbers. Some equivalents of s-numbers are Kolmogorov numbers, Weyl numbers and approximation numbers. Pietsch defined in [1] the concept of s-number sequence to combine all s-numbers in one definition. After some revisions on this definition s-number sequence is presented in [2], [3].

In this paper, by $\mathbb{N}$ and $\mathbb{R}^+$ we denote the set of all natural numbers and nonnegative real numbers, respectively.

A finite rank operator is a bounded linear operator whose dimension of the range space is finite [4].

Let $X$ and $Y$ be real or complex Banach spaces. The space of all bounded linear operators from $X$ to $Y$ and the space of all bounded linear operators between any two arbitrary Banach spaces are denoted by $\mathcal{L}(X,Y)$ and $\mathcal{L}$, respectively.

An s-number sequence is a map $s = (s_n): \mathcal{L} \rightarrow \mathbb{R}^+$ which assigns every operator $T \in \mathcal{L}$ to a nonnegative scalar sequence $(s_n(T))_{n \in \mathbb{N}}$ if the following conditions hold for all Banach spaces $X,Y,X_0$ and $Y_0$:

(S1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for every $T \in \mathcal{L}(X,Y)$,
(S2) \( s_{m+n-1}(S + T) \leq s_m(T) + s_n(T) \) for every \( S, T \in \mathcal{L}(X,Y) \) and \( m, n \in \mathbb{N} \).

(S3) \( s_n(RST) \leq \|R\|s_n(S)\|T\| \) for some \( R \in \mathcal{L}(Y,Y_0) \), \( S \in \mathcal{L}(X,Y) \) and \( T \in \mathcal{L}(X_0,X) \), where \( X_0, Y_0 \) are arbitrary Banach spaces.

(S4) If \( \text{rank}(T) \leq n \), then \( s_n(T) = 0 \).

(S5) \( s_n(I: l^n_1 \rightarrow l^n_2) = 1 \), where \( I \) denotes the identity operator on the \( n \)-dimensional Hilbert space \( l^n_2 \), where \( s_n(T) \) denotes the \( n \)-th s-number of the operator \( T \) [5].

As an example of s-numbers \( a_n(T) \), the \( n \)-th approximation number, is defined as
\[
a_n(T) = \inf\{\|T - A\| : A \in \mathcal{L}(X,Y), \text{rank}(A) < n\},
\]
where \( T \in \mathcal{L}(X,Y) \) and \( n \in \mathbb{N} \) [6].

Let \( T \in \mathcal{L}(X,Y) \) and \( n \in \mathbb{N} \). The other examples of s-number sequences are given in the following, namely Gelfand number \((c_n(T))\), Kolmogorov number \((d_n(T))\), Weyl number \((\chi_n(T))\), Chang number \((y_n(T))\), Hilbert number \((h_n(T))\), etc. For the definitions of these sequences we refer to [4], [7]. In the sequel there are some properties of s-number sequences.

When any metric injection \( J \in \mathcal{L}(Y,Y_0) \) is given and an s-number sequence \( s = (s_n) \) satisfies \( s_n(T) = s_n(JT) \) for all \( T \in \mathcal{L}(X,Y) \) the s-number sequence is called injective [3].

**Proposition 1.** The number sequences \((c_n(T))\) and \((x_n(T))\) are injective [3].

When any metric surjection \( S \in \mathcal{L}(X_0,X) \) is given and an s-number sequence \( s = (s_n) \) satisfies \( s_n(T) = s_n(TS) \) for all \( T \in \mathcal{L}(X,Y) \) the s-number sequence is called surjective [3].

**Proposition 2.** The number sequences \((d_n(T))\) and \((y_n(T))\) are surjective [3].

**Proposition 3.** Let \( T \in \mathcal{L}(X,Y) \). Then \( h_n(T) \leq x_n(T) \leq c_n(T) \leq a_n(T) \) and \( h_n(T) \leq y_n(T) \leq d_n(T) \leq a_n(T) \) [3].

**Lemma 1.** Let \( S, T \in \mathcal{L}(X,Y) \), then \( |s_n(T) - s_n(S)| \leq \|T - S\| \) for \( n = 1,2, \cdots \) [1].

A sequence space is defined as any vector subspace of \( \omega \), where \( \omega \) is the space of real valued sequences.

The Cesaro sequence space \( ces_p \) is defined as
\[
\text{ces}_p = \{ x = (x_k) \in \omega : \sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=1}^n |x_k| \right)^p < \infty \}
\]
where \( 1 < p < \infty \) [8], [9], [10].

If an operator \( T \in \mathcal{L}(X,Y) \) satisfies \( \sum_{n=1}^\infty (a_n(T))^p < \infty \) for \( 0 < p \leq \infty \), it is defined as an \( l_p \) type operator in [6] by Pietsch. Then Constantin defined a new class named ces-p type operators, via Cesaro sequence spaces. If an operator \( T \in \mathcal{L}(X,Y) \) satisfies \( \sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=1}^n a_n(T) \right)^p < \infty \), \( 1 < p < \infty \), it is called ces-p type operator. The class of ces-p type operators includes the class of \( l_p \) type operators [11]. Later on Tita [12] proved that the class of \( l_p \) type operators and ces-p type operators are coincides. Some other generalizations of \( l_p \) type operators were examined in [4], [13][14], [15].

Continuous linear functionals on \( X \) are compose the dual of \( X \) which is denoted by \( X' \). Let \( x' \in X' \) and \( y \in Y \), then the map \( x' \otimes y : X \rightarrow Y \) is defined by
\[
(x'(\otimes y))(x) = x'(x)y, x \in X.
\]
A subcollection \( \mathfrak{S} \) of \( \mathcal{L} \) is called an operator ideal if every component \( \mathfrak{S}(X,Y) = \mathfrak{S} \cap \mathcal{L}(X,Y) \) satisfies the following conditions:

**i)** if \( x' \in X' \), \( y \in Y \), then \( x' \otimes y \in \mathfrak{S}(X,Y) \),

**ii)** if \( S, T \in \mathfrak{S}(X,Y) \), then \( S + T \in \mathfrak{S}(X,Y) \),

**iii)** if \( S \in \mathfrak{S}(X,Y) \), \( T \in \mathcal{L}(X_0,X) \) and \( R \in \mathcal{L}(Y,Y_0) \), then \( RST \in \mathfrak{S}(X_0,Y_0) \) [2].

Let \( \mathfrak{S} \) be an operator ideal and \( \alpha : \mathfrak{S} \rightarrow \mathbb{R}^+ \) be a function on \( \mathfrak{S} \). Then, if the following conditions satisfied:

**i)** If \( x' \in X' \), \( y \in Y \), then \( \alpha(x'(\otimes y)) = \|x'\|\|y\| \),
there exists a constant $c \geq 1$ such that $\alpha(S + T) \leq c[\alpha(S) + \alpha(T)]$,

if $S \in \mathfrak{S}(X, Y), T \in \mathcal{L}(X_0, X)$ and $R \in \mathcal{L}(Y, Y_0)$, then $\alpha(RST) \leq \|R\| \alpha(S)\|T\|$.

$\alpha$ is called a quasi-norm on the operator ideal $\mathfrak{S}$ [2].

For special case $c = 1$, $\alpha$ is a norm on the operator ideal $\mathfrak{S}$.

If $\alpha$ is a quasi-norm on an operator ideal $\mathfrak{S}$, it is denoted by $[3, \alpha]$. Also if every component $\mathfrak{S}(X, Y)$ is complete with respect to the quasinorm $\alpha$, $[3, \alpha]$ is called a quasi-Banach operator ideal.

Let $[3, \alpha]$ be a quasi-normed operator ideal and $J \in \mathcal{L}(Y, Y_0)$ be a metric injection. If for every operator $T \in \mathfrak{S}(X, Y)$ and $J \in \mathfrak{S}(X, Y_0)$ we have $T \in \mathfrak{S}(X, Y)$ and $\alpha(JT) = \alpha(T)$, $[3, \alpha]$ is called an injective quasi-normed operator ideal. Furthermore, let $[3, \alpha]$ be a quasi-normed operator ideal and $Q \in \mathcal{L}(X_0, X)$ be a metric surjection. If for every operator $T \in \mathfrak{S}(X, Y)$ and $TQ \in \mathfrak{S}(X_0, X)$ we have $T \in \mathfrak{S}(X, Y)$ and $\alpha(TQ) = \alpha(T)$, $[3, \alpha]$ is called a surjective quasi-normed operator ideal [2].

Let $T'$ be the dual of $T$. An s-number sequence is called symmetric if $s_n(T) \geq s_n(T')$ for all $T \in \mathcal{L}$. If $s_n(T) = s_n(T')$ the s-number sequence is said to be completely symmetric [2].

For every operator ideal $\mathfrak{S}$, the dual operator ideal denoted by $\mathfrak{S}'$ is defined as $\mathfrak{S}'(X, Y) = \{T \in \mathcal{L}(X, Y): T' \in \mathfrak{S}(Y', X')\}$, where $T'$ is the dual of $T$ and $X'$ and $Y'$ are the duals of $X$ and $Y$, respectively.

An operator ideal $\mathfrak{S}$ is called symmetric if $\mathfrak{S} \subset \mathfrak{S}'$ and is called completely symmetric if $\mathfrak{S} = \mathfrak{S}'$ [2].

Let $E = (E_n)$ be a partition of finite subsets of positive integers such that $\max E_n < \max E_{n+1}$ for $n = 1, 2, \cdots$. Foroutannia, in [16] defined the sequence space $l_p(E)$ as

$$l_p(E) = \left\{ x = (x_n) \in \omega: \sum_{n=1}^{\infty} \sum_{j \in E_n} x_j^p < \infty \right\},$$

where $1 \leq p < \infty$ with the seminorm $\|x\|_{p,E}$ which defined in the following way:

$$\|x\|_{p,E} = \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} x_j \right)^p \right)^{\frac{1}{p}}.$$

For example if $E_n = \{2n - 1, 2n\}$ for all $n$, then $x = (x_n) \in l_p(E)$ if and only if $\sum_{n=1}^{\infty} |x_{2n-1} + x_{2n}|^p < \infty$. It is obvious that $\|x\|_{p,E}$ cannot be a norm, since if $x = (1, -1, 00, \cdots)$ and $E_n = \{2n - 1, 2n\}$ for all $n$ then $\|k\|_{p,E} = 0$ while $x \neq \theta$. In the special case $E_n = \{n\}$ for $n = 1, 2, \cdots$, we have $l_p(E) = l_p$ and $\|x\|_{p,\mathbb{E}} = \|x\|_p$.

For more information about block sequence spaces we refer to [17], [18].

2. MAIN RESULTS

Let $u = (u_n)$ and $v = (v_n)$ be positive real number sequences. In this section we give the definition of the sequence space $X(u, v; l_p(E))$ as follows:

$$X(u, v; l_p(E)) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j x_j(T) \right)^p < \infty \right\}.$$

An operator $T \in \mathcal{L}(X, Y)$ is in the class of $L_{u,v;E}(X, Y)$ if

$$\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j x_j(T) \right)^p < \infty, \quad (1 \leq p < \infty)$$

The class of all s-type $X(u, v; l_p(E))$ operators are denoted by $L_{u,v;E}$. 
Theorem 1. The class $L_{u,v,E}$ is an operator ideal for $1 \leq p < \infty$ where $v_{2k-1} + v_{2k} \leq Mv_k$ $(M > 0)$ and $\sum_{n=1}^{\infty} (u_n)^p < \infty$.

Proof.

$$\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} u_n v_j s_j(x \otimes y) \right)^p = \left( \sum_{j \in E_n} u_n v_j s_j(x \otimes y) \right)^p$$

$$= u_1^p v_1^p \|x \otimes y\|^p$$

$$= u_1^p v_1^p \|x \otimes y\|^p < \infty$$

Since the rank of the operator $x \otimes y$ is one, $s_n(x \otimes y) = 0$ for $n \geq 2$. Therefore $x \otimes y \in L_{u,v,E}$.

Let $S, T \in L_{u,v,E}$. Then

$$\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} u_n v_j s_j(S) \right)^p < \infty, \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j s_j(T) \right)^p < \infty$$

To show that $S + T \in L_{u,v,E}(X,Y)$, we begin with

$$\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} u_n v_j s_j(S + T) \right)^p$$

$$\leq \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} u_n v_{2j-1} s_{2j-1}(S + T) \right)^p + \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} u_n v_{2j} s_{2j}(S + T) \right)^p$$

$$\leq \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} \left( s_{2j-1} + s_{2j} \right)(S + T) \right)^p$$

$$\leq \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} s_j(S + T) \right)^p$$

By using Minkowski inequality;

$$\left( \sum_{j \in E_n} \left( s_j(S + T) \right) \right)^p$$

$$\leq \sum_{j \in E_n} v_j s_j(S) + \sum_{j \in E_n} v_j s_j(T)$$

$$\leq M \left( \sum_{j \in E_n} v_j s_j(S) \right)^{\frac{1}{p}}$$

$$+ M \left( \sum_{j \in E_n} v_j s_j(T) \right)^{\frac{1}{p}} < \infty$$

Hence $S + T \in L_{u,v,E}(X,Y)$.

Let $R \in \mathcal{L}(Y_0, X), S \in L_{u,v,E}(X,Y)$ and $T \in \mathcal{L}(X_0, X)$

$$\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} u_n v_j s_j(RST) \right)^p \leq \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} \|R\|\|T\|v_j s_j(S) \right)^p$$

$$\leq \|R\|^p \|T\|^p \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} u_n v_j s_j(S) \right)^p \right)^{\frac{1}{p}} < \infty$$

So $RST \in L_{u,v,E}(X_0, Y_0)$.

Therefore $L_{u,v,E}$ is an operator ideal.

Theorem 2. $\|T\|_{u,v,E} = \frac{\left( \sum_{n=1}^{\infty} (u_n \sum_{j \in E_n} v_j s_j(T))^p \right)^{\frac{1}{p}}}{u_1 v_1}$ is a quasi-norm on the operator ideal $L_{u,v,E}$.

Proof.

$$\left( \sum_{n=1}^{\infty} (u_n \sum_{j \in E_n} v_j s_j(x \otimes y))^p \right)^{\frac{1}{p}}$$

$$= \frac{(u_1^p v_1^p \|x \otimes y\|^p)^{\frac{1}{p}}}{u_1 v_1} = \|x \otimes y\|_E$$

Since rank of the operator $x \otimes y$ is one, $s_n(x \otimes y) = 0$ for $n \geq 2$. Therefore $\|x \otimes y\|_{u,v,E} = \|x \otimes y\|_E$.

Let $S, T \in L_{u,v,E}$. Then

$$\sum_{j \in E_n} v_j s_j(S + T) \leq \sum_{j \in E_n} v_{2j-1} s_{2j-1}(S + T) + \sum_{j \in E_n} v_{2j} s_{2j}(S + T)$$

$$\leq \sum_{j \in E_n} (v_{2j-1} + v_{2j}) s_{2j-1}(S + T)$$

$$\leq M \sum_{j \in E_n} v_j (s_j(S) + s_j(T))$$
By using Minkowski inequality;
\[
\left(\sum_{n=1}^{\infty} \left(\sum_{j \in E_n} u_n v_j s_j(S + T)\right)^p\right)^{\frac{1}{p}} \leq M \left(\sum_{n=1}^{\infty} \left(\sum_{j \in E_n} u_n v_j s_j(S)\right)^p\right)^{\frac{1}{p}} + \frac{1}{p} \leq \frac{1}{p} \left(\sum_{n=1}^{\infty} \left(\sum_{j \in E_n} u_n v_j s_j(S)\right)^p\right)^{\frac{1}{p}} < \infty
\]
Therefore \(\|S + T\|_{u,v;E} \leq M(\|S\|_{u,v;E} + \|T\|_{u,v;E})\).

Let \(R \in \mathcal{L}(Y, Y_0), S \in L_{u,v;E}(X, Y)\) and \(T \in \mathcal{L}(X, Y)\)
\[
\left(\sum_{n=1}^{\infty} \left(\sum_{j \in E_n} u_n v_j s_j(RST)\right)^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} \left(\sum_{j \in E_n} \|R\|\|T\|v_j s_j(S)\right)^p\right)^{\frac{1}{p}} \leq \|R\|\|T\| \left(\sum_{n=1}^{\infty} \left(\sum_{j \in E_n} u_n v_j s_j(S)\right)^p\right)^{\frac{1}{p}} < \infty
\]
Hence \(\|RST\|_{u,v;E} \leq \|R\|\|T\|\|S\|_{u,v;E}\).
Therefore \(\|T\|_{u,v;E}\) is a quasi-norm on \(L_{u,v;E}\).

**Theorem 3.** Let \(1 \leq p < \infty, [L_{u,v;E}, \|T\|_{u,v;E}]\) is a quasi-Banach operator ideal.

**Proof:** Let \(X, Y\) be any two Banach spaces and \(1 \leq p < \infty\). The following inequality holds
\[
\|T\|_{u,v;E} = \left(\sum_{n=1}^{\infty} \left(\sum_{i \in E_n} u_n v_i s_i(T)\right)^p\right)^{\frac{1}{p}} \geq \|T\|
\]
for \(T \in L_{u,v;E}\).

Let \((T_m)\) be a Cauchy in \(L_{u,v;E}(X, Y)\). Then for every \(\varepsilon > 0\) there exists \(n_0 \in \mathbb{N}\) such that
\[
\|T_m - T_l\|_{u,v;E} < \varepsilon
\]
(2.1)
For all \(m, l \geq n_0\). It follows that
\[
\|T_m - T_l\| \leq \|T_m - T_l\|_{u,v;E} < \varepsilon.
\]
Then \((T_m)\) is a Cauchy sequence in \(\mathcal{L}(X, Y)\). \(\mathcal{L}(X, Y)\) is a Banach space since \(Y\) is a Banach space. Therefore \(\|T_m - T\| \rightarrow 0\) as \(m \rightarrow \infty\) for \(T \in \mathcal{L}(X, Y)\). Now we show that \(\|T_m - T\|_{u,v;E} \rightarrow 0\) as \(m \rightarrow \infty\) for \(T \in L_{u,v;E}(X, Y)\).

The operators \(T_l - T_m, T - T_m\) are in the class \(\mathcal{L}(X, Y)\) for \(T_m, T_l, T \in \mathcal{L}(X, Y)\).
\[
s_n(T_l - T_m) - s_n(T - T_m) \leq \|T_l - T_m - (T - T_m)\|
\]
(2.2)
Since \(T \rightarrow T\) as \(l \rightarrow \infty\) that is \(\|T_l - T\| < \varepsilon\) we obtain
\[
s_n(T_l - T_m) \rightarrow s_n(T - T_m) \text{ as } l \rightarrow \infty.
\]
It follows from (2.1) that the statement
\[
\|T_m - T_l\|_{u,v;E} = \left(\sum_{n=1}^{\infty} \left(\sum_{i \in E_n} u_n v_i s_i(T_m - T_l)\right)^p\right)^{\frac{1}{p}} \leq \varepsilon
\]
holds for all \(m, l \geq n_0\). We obtain from (2.2) that
\[
\left(\sum_{n=1}^{\infty} \left(\sum_{i \in E_n} u_n v_i s_i(T_m - T)\right)^p\right)^{\frac{1}{p}} \leq \varepsilon
\]
as \(l \rightarrow \infty\).
Hence we have \(\|T_m - T\|_{u,v;E} < \varepsilon\) for all \(m \geq n_0\).
Finally we show that \(T \in L_{u,v;E}(X, Y)\),
\[
\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j \mu_j(T) \right)^p \\
\leq \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j \mu_j(T) \right)^p \\
+ \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j \mu_j(T) \right)^p \\
\leq M \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j \mu_j(T) \right)^p \right)^{\frac{1}{p}} \\
+ M \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j \mu_j(T) \right)^q \right)^{\frac{1}{q}} < \infty
\]

By using Minkowski inequality; since \( T_n \in L_{u,v,E}(X,Y) \) for all \( m \) and \( \|T_m - T\|_{u,v,E} \to 0 \) as \( m \to \infty \), we have

\[
M \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j (s_j(T-T_m) + s_j(T_m)) \right)^p \right)^{\frac{1}{p}} \\
\leq M \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j (s_j(T-T_m)) \right)^p \right)^{\frac{1}{p}} \\
+ M \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j (s_j(T_m)) \right)^p \right)^{\frac{1}{p}} < \infty
\]

which means that \( \in L_{u,v,E}(X,Y) \).

**Proposition 1.** The inclusion \( L^p_{u,v,E} \subseteq L^q_{u,v,E} \) holds for \( 1 < p \leq q < \infty \).

**Proof:** Since \( l_p \subseteq l_q \) for \( 1 < p \leq q < \infty \) we have \( L^p_{u,v,E} \subseteq L^q_{u,v,E} \).

Let \( \mu = \left( \mu_n(T) \right) \) be one of the sequences \( a = (a_n(T)), \ c = (c_n(T)), \ d = (d_n(T)), \ x = (x_n(T)), \ y = (y_n(T)) \) and \( h = (h_n(T)) \). Then we define the space \( L^p_{u,v,E}(X,Y) \) and the norm \( \|T\|_{u,v,E}^{(p)} \) as follows:

\[
L^p_{u,v,E}(X,Y) = \left\{ T \in L(X,Y) : \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j \mu_j(T) \right)^p < \infty, \quad (1 < p \leq \infty) \right\}
\]

and

\[
\|T\|_{u,v,E}^{(p)} = \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j \mu_j(T) \right)^p \right)^{\frac{1}{p}}
\]

**Theorem 4.** Let \( 1 < p < \infty \). The quasi-Banach operator ideal \( L^{(s)}_{u,v,E}, \|T\|_{u,v,E}^{(s)} \) is injective, if \( s \)-number sequence is injective.

**Proof:** Let \( 1 < p < \infty \) and \( T \in L(X,Y) \) and \( I \in \mathcal{I}(Y,Y_0) \) be any metric injection. Suppose that \( IT \in L^{(s)}_{u,v,E}(X,Y_0) \). Then

\[
\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j s_j(IT) \right)^p < \infty
\]

Since \( s = (s_n) \) is injective, we have

\[
s_n(T) = s_n(IT) \quad \text{for all} \quad T \in L(X,Y), \quad n = 1,2, \ldots.
\]

(2.3)

Hence we get

\[
\sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j s_j(T) \right)^p < \infty
\]

Thus \( T \in L^{(s)}_{u,v,E}(X,Y) \) and we have from (2.3)

\[
\|IT\|_{u,v,E}^{(s)} = \left( \sum_{n=1}^{\infty} \left( \sum_{j \in E_n} v_j s_j(IT) \right)^p \right)^{\frac{1}{p}}
\]

Hence the operator ideal \( L^{(s)}_{u,v,E}, \|T\|_{u,v,E}^{(s)} \) is injective.

**Corollary 1.** Since the number sequences \( (c_n(T)) \) and \( (x_n(T)) \) are injective, the quasi-
Banach operator ideals \( \left[ L_{u,v,E}^{(c)} \right| \|T\|_{u,v,E}^{(c)} \) and 
\( \left[ L_{u,v,E}^{(x)} \right| \|T\|_{u,v,E}^{(x)} \) are injective [3].

**Theorem 5.** Let \( 1 < p < \infty \). The quasi-Banach operator ideal 
\( \left[ L_{u,v,E}^{(s)} \right| \|T\|_{u,v,E}^{(s)} \) is surjective, if s-number sequence is surjective.

**Proof.** Let \( 1 < p < \infty \) and \( T \in \mathcal{L}(X,Y) \) and \( S \in \mathcal{L}(X, Y) \) be any metric injection. Suppose that 
\( TS \in L_{u,v,E}^{(s)}(X, Y) \). Then

\[
\sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j s_j(TS) \right)^p < \infty.
\]

Since \( s = (s_n) \) is surjective, we have 
\( s_n(T) = s_n(TS) \) for all \( T \in \mathcal{L}(X, Y) \), \( n = 1, 2, ..., \) (2.4)

Hence we get

\[
\sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j s_j(TS) \right)^p = \sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j S_j(TS) \right)^p < \infty.
\]

Thus \( T \in L_{u,v,E}^{(s)}(X, Y) \) and we have from (2.4)

\[
\|TS\|_{u,v,E}^{(s)} = \left( \frac{\sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j S_j(TS) \right)^p}{\left( \sum_{n=1}^{\infty} (u_n)^p \right)^{\frac{1}{p}} v_1} \right)^{\frac{1}{p}} = \|T\|_{u,v,E}^{(s)}.
\]

Hence the operator ideal \( \left[ L_{u,v,E}^{(s)} \right| \|T\|_{u,v,E}^{(s)} \) is surjective.

**Corollary 2.** Since the number sequences \( (d_n(T)) \) and \( (\Omega_n(T)) \) are surjective, the quasi-Banach operator ideals 
\( \left[ L_{u,v,E}^{(d)} \right| \|T\|_{u,v,E}^{(d)} \) and \( \left[ L_{u,v,E}^{(\Omega)} \right| \|T\|_{u,v,E}^{(\Omega)} \) are surjective [3].

**Theorem 6.** Let \( 1 < p < \infty \). Then the following inclusion relations hold:

i. \( L_{u,v,E}^{(a)} \subseteq L_{u,v,E}^{(c)} \subseteq L_{u,v,E}^{(x)} \subseteq L_{u,v,E}^{(h)} \)

ii. \( L_{u,v,E}^{(a)} \subseteq L_{u,v,E}^{(d)} \subseteq L_{u,v,E}^{(\Omega)} \subseteq L_{u,v,E}^{(h)} \).

**Proof.** Let \( T \in L_{u,v,E}^{(a)} \). Then

\[
\sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j s_j(T) \right)^p < \infty,
\]

where \( 1 < p < \infty \). And from Proposition 3, we have;

\[
\sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j h_j(T) \right)^p \leq \sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j x_j(T) \right)^p \leq \sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j c_j(T) \right)^p < \infty,
\]

and

\[
\sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j s_j(T) \right)^p \leq \sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j y_j(T) \right)^p \leq \sum_{n=1}^{\infty} \left( u_n \sum_{j \in E_n} v_j d_j(T) \right)^p < \infty.
\]

So it is shown that the inclusion relations are satisfied.

**Theorem 7.** The operator ideal \( L_{u,v,E}^{(a)} \) is symmetric and the operator ideal \( L_{u,v,E}^{(h)} \) is completely symmetric for \( 1 < p < \infty \).

**Proof.** Let \( 1 < p < \infty \).
Firstly, we prove that the inclusion $l^{(a)}_{u,v,E} \subseteq \left( l^{(a)}_{u,v,E} \right)'$ holds. Let $T \in l^{(a)}_{u,v,E}$. Then

$$
\sum_{n=1}^{\infty} \left( u_n \sum_{j \in \mathbb{E}_n} v_j a_j(T) \right)^p < \infty.
$$

It follows from [2] $a_n(T') \leq a_n(T)$ for $T \in \mathcal{L}$. Hence we get

$$
\sum_{n=1}^{\infty} \left( u_n \sum_{j \in \mathbb{E}_n} v_j a_j(T') \right)^p \leq \sum_{n=1}^{\infty} \left( u_n \sum_{j \in \mathbb{E}_n} v_j a_j(T) \right)^p < \infty.
$$

Therefore $T \in \left( l^{(a)}_{u,v,E} \right)'$. Thus $l^{(a)}_{u,v,E}$ is symmetric.

Now we prove that the equation $L^{(h)}_{u,v,E} = \left( L^{(h)}_{u,v,E} \right)'$ holds. It follows from [3] that $h_n(T') = h_n(T)$ for $T \in \mathcal{L}$. Then we can write

$$
\sum_{n=1}^{\infty} \left( u_n \sum_{j \in \mathbb{E}_n} v_j h_j(T') \right)^p \leq \sum_{n=1}^{\infty} \left( u_n \sum_{j \in \mathbb{E}_n} v_j h_j(T) \right)^p.
$$

Hence $L^{(h)}_{u,v,E}$ is completely symmetric.

**Theorem 8** Let $1 < p < \infty$. The equation $L^{(c)}_{u,v,E} = \left( L^{(d)}_{u,v,E} \right)'$ and the inclusion relation $L^{(d)}_{u,v,E} \subseteq \left( L^{(c)}_{u,v,E} \right)'$ holds. Also, the equation $L^{(d)}_{u,v,E} = \left( L^{(c)}_{u,v,E} \right)'$ holds for any compact operators.

**Proof:** Let $1 < p < \infty$. For $T \in \mathcal{L}$ we have from [3] that $c_n(T) = d_n(T')$ and $c_n(T') \leq d_n(T)$. Also, if $T$ is a compact operator, then the equality $c_n(T') = d_n(T)$ holds. Thus the proof is clear.

**Theorem 9** $L^{(x)}_{u,v,E} = \left( L^{(y)}_{u,v,E} \right)'$ and $L^{(y)}_{u,v,E} = \left( L^{(x)}_{u,v,E} \right)'$ hold.

**Proof:** Let $1 < p < \infty$. For $T \in \mathcal{L}$ we have from [3] that $x_n(T') = y_n(T')$ and $y_n(T) = x_n(T')$. Thus the proof is clear.

### 3. REFERENCES


