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Some New Inequalities for \((\alpha, m_1, m_2)\)-GA Convex Functions

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Abstract

In this manuscript, firstly we introduce and study the concept of \((\alpha, m_1, m_2)\)-Geometric-Arithmetically (GA) convex functions and some algebraic properties of such type functions. Then, we obtain Hermite-Hadamard type integral inequalities for the newly introduced class of functions by using an identity together with Hölder integral inequality, power-mean integral inequality and Hölder-İşcan integral inequality giving a better approach than Hölder integral inequality. Inequalities have been obtained with the help of Gamma function. In addition, results were obtained according to the special cases of \(\alpha, m_1\) and \(m_2\).

Keywords: \((\alpha, m_1, m_2)\)-GA convex function, Hölder integral inequality, power-mean inequality, Hölder-İşcan inequality, Hermite-Hadamard integral inequality.

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1. INTRODUCTION

Let \( f: I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). Then the following inequalities

\[
 f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a)+f(b)}{2}
\]

hold. Both inequalities hold in the reversed direction if the function \( f \) is concave \([4, 6]\). The above inequalities were firstly discovered by the famous scientist Charles Hermite. This double inequality is well-known in the literature as Hermite-Hadamard integral inequality for convex functions. This inequality gives us upper and lower bounds for the integral mean-value of a convex function. Some of the classical inequalities for means can be derived from Hermite-Hadamard inequality for appropriate particular selections of the function \( f \).

Convexity theory plays an important role in mathematics and many other sciences. It provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. Readers can find more information in the recent studies \([1, 5, 8, 10, 11, 15, 19, 20, 23, 24, 25]\) and the references therein for different convex classes and related Hermite-Hadamard integral inequalities.

**Definition 1.** ([17,18]) A function \( f: I \subseteq \mathbb{R}_{+} = (0, \infty) \rightarrow \mathbb{R} \) is said to be GA-convex function on \( I \) if the inequality

\[
f(x^\lambda y^{1-\lambda}) \leq \lambda f(x) + (1-\lambda)f(y)
\]

holds for all \( x,y \in I \) and \( \lambda \in [0,1] \), where \( x^\lambda y^{1-\lambda} \) and \( \lambda f(x) + (1-\lambda)f(y) \) are respectively the weighted geometric mean of two positive numbers \( x \) and \( y \) and the weighted arithmetic mean of \( f(x) \) and \( f(y) \).

**Definition 2.** ([22]) A function \( f: [0, b] \rightarrow \mathbb{R} \) is said to be \( m \)-convex for \( m \in (0,1) \) if the inequality

\[
f(ax + m(1-\alpha)y) \leq \alpha f(x) + m(1-\alpha)f(y)
\]

holds for all \( x,y \in [0, b] \) and \( \alpha \in [0,1] \).

**Definition 3.** ([12]) The function \( f: [0, b] \rightarrow \mathbb{R}, b > 0 \), is said to be \((m_1, m_2)\)-convex, if the inequality

\[
f(m_1tx + m_2(1-t)y) \leq m_1 tf(x) + m_2(1-t)f(y)
\]

holds for all \( x,y \in I, t \in [0,1] \) and \((m_1, m_2) \in (0,1)^2\).

**Definition 4.** ([13]) \( f: [0, b] \rightarrow \mathbb{R}, b > 0 \), is said to be \((\alpha, m_1, m_2)\)-convex function, if the inequality

\[
f(m_1tx + m_2(1-t)y) \leq m_1 t^\alpha f(x) + m_2(1-t)^\alpha f(y)
\]

holds for all \( x,y \in I, t \in [0,1] \) and \((\alpha, m_1, m_2) \in (0,1)^3\).

**Definition 5.** ([16]) For \( f: [0, b] \rightarrow \mathbb{R} \) and \((\alpha, m) \in (0,1)^2\), if

\[
f(tx + (1-t)y) \leq t^\alpha f(x) + m(1-t)^\alpha f(y)
\]

is valid for all \( x,y \in [0, b] \) and \( t \in [0,1] \), then we say that \( f(x) \) is an \((\alpha, m)\)-convex function on \([0, b]\).

**Definition 6.** ([17]) The GG-convex functions (called in what follows multiplicatively convex functions) are those functions \( f: I \rightarrow I \) (acting on subintervals of \((0, \infty)\)) such that

\[
x,y \in I \text{ and } \lambda \in [0,1] \Rightarrow f(x^{1-t}y^t) \leq f(x)^{1-\lambda} f(y)^\lambda
\]

i.e., it is called log-convexity and it is different from the above.

**Definition 7.** ([9]) Let the function \( f: [0, b] \rightarrow \mathbb{R} \) and \((\alpha, m) \in (0,1)^2\). If

\[
f(x^t y^{m(1-t)}) \leq t^\alpha f(a) + m(1-t)^\alpha f(b). \]  \hspace{1cm} (1.1)

for all \([a, b] \in [0, b] \) and \( t \in [0,1] \), then \( f(x) \) is said to be \((\alpha, m)\)-geometric arithmetically convex function or, simply speaking, an \((\alpha, m)\)-GA-convex function. If (1.1) reversed, then \( f(x) \) is
said to be \((\alpha, m)\)-geometric arithmetically concave function or, simply speaking, an \((\alpha, m)\)-GA-concave function.

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

**Theorem 1.** (Hölder-İşcan integral inequality [7])

Let \(p > 1\) and \(\frac{1}{p} + \frac{1}{q} = 1\). If \(f\) and \(g\) are real functions defined on \([a, b]\) and if \(|f|^p\), \(|g|^q\) are integrable functions on the interval \([a, b]\) then

\[
\int_{a}^{b} |f(x)g(x)|dx \\
\leq \frac{1}{b-a} \left\{ \left[ \int_{a}^{b} |f(x)|^pdx \right]^\frac{1}{p} \left[ \int_{a}^{b} |g(x)|^qdx \right]^\frac{1}{q} + \left( \int_{a}^{b} |f(x)|^pdx \right)^\frac{1}{p} \left( \int_{a}^{b} |g(x)|^qdx \right)^\frac{1}{q} \right\}.
\]

**Definition 8.** (Gamma function)

The classic gamma function is usually defined for \(Rez > 0\) by

\[
\Gamma(z) = \int_{0}^{\infty} t^{z-1}e^{-t}dt.
\]

The main purpose of this paper is to introduce the concept of \((\alpha, m_1, m_2)\)-geometric arithmetically convex (GA) convex functions and establish some results connected with new inequalities similar to the Hermite-Hadamard integral inequality for these classes of functions.

**2. MAIN RESULTS FOR \((\alpha, m_1, m_2)\)-GA CONVEX FUNCTIONS**

In this section, we introduce a new concept, which is called \((\alpha, m_1, m_2)\)-GA convex functions and we give by setting some algebraic properties for the \((\alpha, m_1, m_2)\)-GA convex functions, as follows:

**Definition 9.** Let the function \(f: [0, b] \rightarrow \mathbb{R}\) and \((\alpha, m_1, m_2) \in (0, 1]^3\). If

\[
f(a^{m_1}t^{m_2}(1-t)) \leq m_1t^\alpha f(a) + m_2(1-t)^\alpha f(b) \quad (2.1)
\]

for all \([a, b] \subseteq [0, b]\) and \(t \in [0, 1]\), then the function \(f\) is said to be \((\alpha, m_1, m_2)\)-geometric arithmetically convex function, if the inequality (2.1) reversed, then the function \(f\) is said to be \((\alpha, m_1, m_2)\)-geometric arithmetically concave function.

**Example 1.** \(f(x) = c, c < 0\) is a \((\alpha, m_1, m_2)\)-geometric arithmetically convex function.

We discuss some connections between the class of the \((\alpha, m_1, m_2)\)-GA convex functions and other classes of generalized convex functions.

**Remark 1.** When \(m_1 = m_2 = \alpha = 1\), the \((\alpha, m_1, m_2)\)-geometric arithmetically convex (concave) function becomes a geometric arithmetically convex (concave) function defined in [17, 18].

**Remark 2.** When \(m_1 = 1, m_2 = m\), the \((\alpha, m_1, m_2)\)-geometric arithmetically convex (concave) function becomes an \((\alpha, m)\)-geometric arithmetically convex (concave) function defined in [9].

**Remark 3.** When \(m_1 = m_2 = 1\) and \(\alpha = s\), the \((\alpha, m_1, m_2)\)-geometric arithmetically convex (concave) function becomes a geometric arithmetically-s convex (concave) function defined in [14].

**Remark 4.** When \(\alpha = 1\), the \((\alpha, m_1, m_2)\)-geometric arithmetically convex (concave) function becomes a \((m_1, m_2)\)-GA convex (concave) function defined in [21].

**Proposition 1.** The function \(f: l \subseteq (0, \infty) \rightarrow \mathbb{R}\) is \((\alpha, m_1, m_2)\)-GA convex function on \(l\) if and only if \(f \circ \exp: lnl \rightarrow \mathbb{R}\) is \((\alpha, m_1, m_2)\)-convex function on the interval \(lnl = \{lnx| x \in l\}\).

**Proof.** \((\Rightarrow)\) Let \(f: l \subseteq (0, \infty) \rightarrow \mathbb{R}\) \((\alpha, m_1, m_2)\)-GA convex function. Then, we write

\[
(f \circ \exp)(m_1 t lna + m_2(1 - t)lnb)
\]
\[ \leq m_1 t^a (f \circ \exp)(\ln a) + m_2 (1 - t^a) (f \circ \exp)(\ln b). \]

From here, we get
\[ f(a^{m_1 t} b^{m_2 (1-t)}) \leq m_1 t^a f(a) + m_2 (1 - t^a) f(b). \]

Hence, the function \( f \circ \exp \) is \((\alpha, m_1, m_2)\)-convex function on the interval \( \ln I \).

\[ (\Leftarrow) \text{ Let } f \circ \exp : \ln I \to \mathbb{R}, \text{ (}\alpha, m_1, m_2\text{)-convex function on the interval } \ln I. \text{ Then, we obtain} \]
\[ f(a^{m_1 t} b^{m_2 (1-t)}) = f(e^{m_1 t \ln a} + z(1-t) \ln b) \]
\[ = (f \circ \exp)(m_1 t \ln a + m_2 (1 - t) \ln b) \]
\[ \leq m_1 t^a f(e^{\ln a}) + m_2 (1 - t^a) f(e^{\ln b}) \]
\[ = m_1 t^a f(a) + m_2 (1 - t^a) f(b), \]

which means that the function \( f(x) \) \((\alpha, m_1, m_2)\)-GA convex function on \( I \).

**Theorem 2.** Let \( f, g : I \subseteq \mathbb{R} \to \mathbb{R} \). If \( f \) and \( g \) are \((\alpha, m_1, m_2)\)-GA convex functions, then \( f + g \) is an \((\alpha, m_1, m_2)\)-GA convex function and \( cf \) is an \((\alpha, m_1, m_2)\)-GA convex function for \( c \in \mathbb{R}_+ \).

**Proof.** Let \( f, g \) be \((\alpha, m_1, m_2)\)-GA convex functions, then
\[ (f + g)(a^{m_1 t} b^{m_2 (1-t)}) \]
\[ = f(a^{m_1 t} b^{m_2 (1-t)}) + g(a^{m_1 t} b^{m_2 (1-t)}) \]
\[ \leq m_1 t^a f(a) + m_2 (1 - t^a) f(b) \]
\[ + m_1 t^a g(a) + m_2 (1 - t^a) g(b) \]
\[ = m_1 t^a (f + g)(a) + m_2 (1 - t^a)(f + g)(b) \]
Let \( f \) be \((\alpha, m_1, m_2)\)-GA convex function and \( c \in \mathbb{R}(c \geq 0) \), then
\[ (cf)(a^{m_1 t} b^{m_2 (1-t)}) \]
\[ \leq c[m_1 t^a f(x) + m_2 (1 - t^a) f(y)] \]
\[ = m_1 t^a(cf)(x) + m_2 (1 - t^a)(cf)(y). \]

This completes the proof of the theorem.

**Theorem 3.** If \( f : I \to J \) is a \((m_1, m_2)\)-GG convex and \( g : J \to \mathbb{R} \) is a \((\alpha, m_1, m_2)\)-GA convex function and nondecreasing, then \( g \circ f : I \to \mathbb{R} \) is a \((\alpha, m_1, m_2)\)-GA convex function.

**Proof.** For \( a, b \in I \) and \( t \in [0,1] \), we get
\[ (g \circ f)(a^{m_1 t} b^{m_2 (1-t)}) \]
\[ = g(f(a^{m_1 t} b^{m_2 (1-t)})) \]
\[ \leq g((f(a))^{m_1 t}[f(b)]^{m_2 (1-t)}) \]
\[ \leq m_1 t^a g(f(x)) + m_2 (1 - t^a) g(f(y)). \]

This completes the proof of the theorem.

**Theorem 4.** Let \( b > 0 \) and \( f_\beta : [a, b] \to \mathbb{R} \) be an arbitrary family of \((\alpha, m_1, m_2)\)-GA convex functions and let \( f(x) = \sup_\beta f_\beta(x) \). If \( J = \{u \in [a, b] : f(u) < \infty\} \) is nonempty, then \( J \) is an interval and \( f \) is an \((\alpha, m_1, m_2)\)-GA convex function on \( J \).

**Proof.** Let \( t \in [0,1] \) and \( x, y \in J \) be arbitrary. Then
\[ f(a^{m_1 t} b^{m_2 (1-t)}) \]
\[ = \sup_\beta f_\beta(a^{m_1 t} b^{m_2 (1-t)}) \]
\[ \leq \sup_\beta [m_1 t^a f_\beta(x) + m_2 (1 - t^a) f_\beta(y)] \]
\[ \leq m_1 t^a \sup_\beta f_\beta(x) + m_2 (1 - t^a) \sup_\beta f_\beta(y) \]
\[ = m_1 t^a f(x) + m_2 (1 - t^a) f(y) < \infty. \]

This shows simultaneously that \( J \) is an interval since it contains every point between any two of its points, and that \( f \) is an \((\alpha, m_1, m_2)\)-GA convex function on \( J \). This completes the proof of the theorem.
Theorem 5. If the function \( f: [a, b] \rightarrow \mathbb{R} \) is an \((a, m_1, m_2)\)-GA convex function then \( f \) is bounded on the interval \([a, b]\).

Proof. Let \( K = \max\{m_1 f(a), m_2 f(b)\} \) and \( x \in [a, b] \) is arbitrary. Then there exists a \( t \in [0,1] \) such that \( x = a m_1 t b m_2 (1-t) \). Thus, since \( m_1 t^a \leq 1 \) and \( m_2 (1 - t^a) \leq 1 \) we have

\[
f(x) = f(a m_1 t b m_2 (1-t)) \leq m_1 t^a f(a) + m_2 (1 - t^a) f(b) \leq 2K = M.
\]

Also, for every \( x \in [a m_1, b m_2] \) there exists a \( \lambda \in \left[ \frac{a m_1}{b m_2}, 1 \right] \) such that \( x = \lambda a m_1 b m_2 \) and \( x = \frac{1}{\sqrt{\lambda a m_1 b m_2}} \). Without loss of generality, we can suppose \( x = \lambda a m_1 b m_2 \). So, we have

\[
f(\sqrt{a m_1 b m_2}) = f\left(\sqrt{\lambda a m_1 b m_2} \right) \leq f(x) f\left(\frac{\sqrt{a m_1 b m_2}}{\lambda}\right).
\]

By using \( M \) as the upper bound, we obtain

\[
f(x) \geq \frac{f^2\left(\sqrt{a m_1 b m_2}\right)}{f\left(\sqrt{a m_1 b m_2}\right)} \geq \frac{f^2\left(\sqrt{a m_1 b m_2}\right)}{M} = m.
\]

This completes the proof of the theorem.

3. HERMITE-HADAMARD INEQUALITY FOR \((a, m_1, m_2)\)-GA CONVEX FUNCTION

This section aims to establish some inequalities of Hermite-Hadamard type integral inequalities for \((a, m_1, m_2)\)-GA convex functions. In this section, we will denote by \( L[a, b] \) the space of (Lebesgue) integrable functions on the interval \([a, b]\).

Theorem 6. Let \( f: [a, b] \rightarrow \mathbb{R} \) be an \((a, m_1, m_2)\)-GA convex function. If \( a < b \) and \( f \in L[a, b] \), then the following Hermite-Hadamard type integral inequalities hold:

\[
f\left(\sqrt{a m_1 b m_2}\right) \leq \frac{1}{\ln b m_2 - \ln a m_1} \int_{a m_1}^{b m_2} f(u) \frac{du}{u} \leq \frac{m_1 f(a)}{a+1} + \frac{m_2 f(b)}{a+1}. \tag{3.1}
\]

Proof. Firstly, from the property of the \((a, m_1, m_2)\)-GA convex function of \( f \), we can write

\[
f\left(\sqrt{a m_1 b m_2}\right) = f\left(\sqrt{a m_1 t b m_2 (1-t)} a m_1 (1-t) b m_2 t\right) \leq f(a m_1 t b m_2 (1-t)) + f(a m_1 (1-t) b m_2 t). \]

Now, if we take integral in the last inequality with respect to \( t \in [0,1] \), we deduce that

\[
f\left(\sqrt{a m_1 b m_2}\right) \leq \frac{1}{2} \int_0^1 f(a m_1 t b m_2 (1-t)) dt + \frac{1}{2} a m_1 \left(\frac{1-t}{1-t^a}\right) f(b) dt
\]

\[
= \frac{1}{2 \ln b m_2 - \ln a m_1} \int_{a m_1}^{b m_2} f(u) \frac{du}{u}
\]

\[
+ \frac{1}{2 \ln b m_2 - \ln a m_1} \int_{a m_1}^{b m_2} f(u) \frac{du}{u}
\]

\[
= \frac{1}{2 \ln b m_2 - \ln a m_1} \int_{a m_1}^{b m_2} f(u) \frac{du}{u}
\]

Secondly, by using the property of the \((a, m_1, m_2)\)-GA convex function of \( f \), if the variable is changed as \( u = a m_1 t b m_2 (1-t) \), then

\[
\frac{1}{\ln b m_2 - \ln a m_1} \int_{a m_1}^{b m_2} f(u) \frac{du}{u}
\]

\[
= \int_0^1 f(a m_1 t b m_2 (1-t)) dt
\]

\[
\leq \int_0^1 [m_1 t^a f(a) + m_2 (1 - t^a) f(b)] dt
\]

\[
= m_1 f(a) \int_0^1 t^a dt + m_2 f(b) \int_0^1 (1 - t^a) dt
\]

\[
= \frac{m_1 f(a)}{a+1} + \frac{m_2 f(b)}{a+1}
\]

This completes the proof of the theorem.
Corollary 1. By considering the conditions of Theorem 6, if we take \( m_1 = m_2 = 1 \) and \( \alpha = 1 \) in the inequality (3.1), then we get
\[
f(\sqrt{ab}) \leq \frac{1}{ln b - ln a} \int_{a}^{b} \frac{f(u)}{u} du \leq \frac{f(a) + f(b)}{2}.
\]
This inequality coincides with the inequality in [2].

Corollary 2. By considering the conditions of Theorem 6, if we take \( \alpha = 1 \) in the inequality (3.1), then we get
\[
f(\sqrt[\alpha]{am_1 b^{m_2}}) \leq \frac{1}{ln b - ln a} \int_{a}^{b} \frac{f(u)}{u} du \leq \frac{m_1 f(a) + m_2 f(b)}{2}.
\]
This inequality coincides with the inequality in [14].

4. SOME NEW INEQUALITIES FOR \((\alpha, m_1, m_2)\)-GA CONVEX FUNCTIONS

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is \((\alpha, m_1, m_2)\)-GA convex function. Ji et al. [9] used the following lemma. Also, we will use this lemma to obtain our results.

Lemma 1. ([3]) Let \( f: I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function and \( a, b \in I \) with \( a < b \). If \( f' \in L([a, b]) \), then
\[
\frac{b^2 f(a) - a^2 f(b)}{2} - \int_{a}^{b} \frac{f(x)}{x} dx \leq \frac{ln b - ln a}{2} \int_{0}^{1} a^{3(1-t)} b^{3t} f'(a^{1-t} b^t) dt.
\]

Theorem 7. Let the function \( f: \mathbb{R}_+ = [0, \infty) \rightarrow \mathbb{R} \) be a differentiable function and \( f' \in L([a, b]) \) for \( 0 < a < b < \infty \). If \( |f'| \) is \((\alpha, m_1, m_2)\)-GA convex on \( [0, \max\{a^{m_1}, b^{m_2}\}] \) for \((\alpha, m_1, m_2) \in (0,1]^3 \), then the following integral inequalities hold
\[
\left| \frac{b^2 f(a) - a^2 f(b)}{2} - \int_{a}^{b} \frac{f(x)}{x} dx \right| \leq \frac{m_1}{2} \left| f' \left( \frac{1}{a^{m_1}} \right) \right| \cdot \frac{b^3 - a^3}{3} \\
+ \frac{m_2}{2} \left| f' \left( \frac{1}{b^{m_2}} \right) \right| \cdot \frac{a^3 - a'^3}{3} \\
= \frac{m_1}{2} \left| f' \left( \frac{1}{a^{m_1}} \right) \left[ \frac{b^3 - a^3}{3} - a^3 \Gamma(\alpha + 1, 3(lna - lnb)) - a^3 \Gamma(a + 1, 0) \right] \right| \\
+ \frac{m_2}{2} \left| f' \left( \frac{1}{b^{m_2}} \right) \left[ \frac{a^3 \Gamma(\alpha + 1, 3(lna - lnb)) - a^3 \Gamma(a + 1, 0)}{3a^3} \right] \right|.
\]

This completes the proof of the theorem.
Corollary 3. By considering the conditions of Theorem 7, if we take $m_1 = m_2 = 1$ and $\alpha = 1$ then we get
\[
\left| \frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b xf(x)dx \right| 
\leq \left| \frac{f'(a)}{6} [L(a^3, b^3) - a^3] + \frac{f'(b)}{6} [b^3 - L(a^3, b^3)] \right|
\]
where $L$ is the logarithmic mean.

Corollary 4. By considering the conditions of Theorem 7, if we take $\alpha = 1$ in the inequality (4.1), then we get
\[
\left| \frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b xf(x)dx \right| 
\leq \frac{m_1}{2} \left| f' \left( \frac{1}{a} \right) \right| [L(a^3, b^3) - a^3] + \frac{m_2}{2} \left| f' \left( \frac{1}{b} \right) \right| [b^3 - L(a^3, b^3)].
\]

Theorem 8. Let the function $f: \mathbb{R} = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$ for $0 < a < b < \infty$. If $|f'|^q$ is $(\alpha, m_1, m_2)$-GA convex on $[0, \max \left\{ \frac{1}{a}, \frac{1}{b} \right\}]$ for $[\alpha, m_1, m_2] \in (0,1]^3$ and $q \geq 1$ then
\[
\left| \frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b xf(x)dx \right| \leq \frac{\ln b - \ln a}{2} L \left\{ \frac{1}{q} \right\} \left[ m_1 \left| f' \left( \frac{1}{a} \right) \right|^q \left( \frac{b^3 - a^3}{3(\ln b - \ln a)} \right) - a^3 f'(a + 1, 0) \right] + \frac{m_2}{2} \left| f' \left( \frac{1}{b} \right) \right|^q \left( \frac{\alpha^3 f'(a + 1, 3(\ln a - \ln b))}{3 a^3 + (\ln a - \ln b)^2 a} \right) \right]^{\frac{1}{q}}
\]
+ $\frac{m_2}{2} \left| f' \left( \frac{1}{b} \right) \right|^q \left( \frac{\alpha^3 f'(a + 1, 3(\ln a - \ln b))}{3 a^3 + (\ln a - \ln b)^2 a} \right) ^{\frac{1}{q}}$,
where $L$ is the logarithmic mean.

Proof. By using both Lemma 1, power-mean inequality and the $(\alpha, m_1, m_2)$-GA convexity of $|f'|^q$ on the interval $[0, \max \left\{ \frac{1}{a}, \frac{1}{b} \right\}]$, that is, the inequality
\[
\left| f'(a^\alpha b^{\alpha_2}) \right| = \left| f' \left( \frac{1}{a} \right) \right|^q \left( \frac{1}{a} \right) f' \left( \frac{1}{b} \right)^{\alpha_2} \right| ^q
\]
\[
\leq m_1 (1 - t^\alpha) \left| f' \left( \frac{1}{a} \right) \right|^q + m_2 t^\alpha \left| f' \left( \frac{1}{b} \right) \right|^q
\]
is satisfied and we get
\[
\left| \frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b xf(x)dx \right| \leq \frac{\ln \left( \frac{b}{a} \right)}{2} \left[ \int_0^1 a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
\left[ \int_0^1 a^3 (1-t) b^{\alpha_2} \left| f' \left( \frac{1}{a} \right) \right|^q \left( \frac{1}{a} \right) f' \left( \frac{1}{b} \right)^{\alpha_2} \right| ^q \int_0^1 t^\alpha a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
= \frac{\ln \left( \frac{b}{a} \right)}{2} \left[ \int_0^1 a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
\cdot \left( \int_0^1 a^3 (1-t) b^{\alpha_2} \left| f' \left( \frac{1}{a} \right) \right|^q \left( \frac{1}{a} \right) f' \left( \frac{1}{b} \right)^{\alpha_2} \right| ^q \int_0^1 t^\alpha a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
= \frac{\ln \left( \frac{b}{a} \right)}{2} \left[ \int_0^1 a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
\cdot \left[ m_1 \left| f' \left( \frac{1}{a} \right) \right|^q \int_0^1 (1 - t^\alpha) a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
+ m_2 \left| f' \left( \frac{1}{b} \right) \right|^q \int_0^1 t^\alpha a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
= \frac{\ln \left( \frac{b}{a} \right)}{2} \left[ \int_0^1 a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
\cdot \left[ m_1 \left| f' \left( \frac{1}{a} \right) \right|^q \left( \frac{1}{a} \right) f' \left( \frac{1}{b} \right)^{\alpha_2} \right| ^q \int_0^1 t^\alpha a^3 (1-t) b^{\alpha_2} dt \right]^{\frac{1}{q}}
\]
\[
+ m_2 \left| f' \left( \frac{1}{b} \right) \right|^q \left( \frac{\alpha^3 f'(a + 1, 3(\ln a - \ln b))}{3 a^3 + (\ln a - \ln b)^2 a} \right) ^{\frac{1}{q}}
\]
This completes the proof of the theorem.
Corollary 5. By considering the conditions of Theorem 8, if we take $m_1 = m_2 = 1$ and $\alpha = 1$ in the inequality (4.2), then we get

$$\left| \frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx \right| \leq \frac{\ln b - \ln a}{2} L \frac{b^3-a^3}{3(a^3,b^3)}$$

$$\times \left[ \left| f'(a) \right|^q \frac{L(a^3,b^3)-b^3}{3(lnb-lna)} + \left| f'(b) \right|^q \frac{b^3-L(a^3,b^3)}{3(lnb-lna)} \right]^\frac{1}{q},$$

where $L$ is the logarithmic mean.

Corollary 8. By considering the conditions of Theorem 8, if we take $q = 1$, then

$$\left| \frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx \right| \leq \left| \frac{m_1}{2} f' \left( \frac{1}{am_1} \right) \left( \frac{b^3-a^3}{3} \right) - \frac{a^3\Gamma(\alpha+1, 3(lna-lnb))-a^3\Gamma(\alpha+1, 0)}{3\alpha+1(lna-lnb)^\alpha} \right| \frac{m_2}{2}$$

$$\times \left| f' \left( \frac{1}{b^m_2} \right) \left( \frac{a^3\Gamma(\alpha+1, 3(lna-lnb))-a^3\Gamma(\alpha+1, 0)}{3\alpha+1(lna-lnb)^\alpha} \right) \right|.$$

This inequality coincides with the inequality (4.1).

Corollary 7. By considering the conditions of Theorem 8, if we take $m_1 = m_2 = 1$ and $\alpha = q = 1$ in the inequality (4.2), then we get

$$\left| \frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx \right| \leq \frac{\ln(b/a)}{2}$$

$$\times \left| \frac{1}{L^p(a^3p,b^3p)} \left\{ \frac{am_1}{\alpha+1} f' \left( \frac{1}{am_1} \right) \right\}^q + \frac{m_2}{\alpha+1} f' \left( \frac{1}{b^m_2} \right) \right\}^\frac{1}{q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using both Lemma 1, Hölder integral inequality and the $(\alpha,m_1,m_2)$-GA-convexity of the function $|f'|^q$ on the interval $[0, \max \{ \alpha m_1, b^m_2 \}]$, that is, the inequality

$$\left| f'(a^{1-t}b^t) \right| = \left| f' \left( \frac{1}{am_1} \right) \frac{m_1}{m_2} \right| = \left| f' \left( \frac{1}{b^m_2} \right) \right|^q,$$

$$\leq m_1(1-t) \left| f' \left( \frac{1}{am_1} \right) \right|^q + m_2t \left| f' \left( \frac{1}{b^m_2} \right) \right|^q,$$

we get

$$\left| \frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx \right| \leq \frac{\ln(b/a)}{2}$$

$$\times \left| \int_0^1 \left( \frac{am_1}{\alpha+1} \right)^{m_1(1-t)} \left( \frac{b^m_2}{\alpha+1} \right)^{m_2t} dt \right|^\frac{1}{q}$$

$$\times \left| \int_0^1 \left( \frac{am_1}{\alpha+1} \right)^{m_1(1-t)} \left( \frac{b^m_2}{\alpha+1} \right)^{m_2t} dt \right|^\frac{1}{q}.$$
Some New Inequalities for \((a,m_1,m_2)\)-GA Convex Functions

\[
\begin{align*}
\leq \frac{\ln(b/a)}{2} \left[ \int_0^1 \left( a^{3(1-t)} b^{3t} \right)^p dt \right]^{\frac{1}{p}} \\
\cdot \left[ \int_0^1 \left[ m_1 (1 - t^\alpha) \left| f' \left( \frac{1}{a^{m_1}} \right) \right|^q + m_2 t^\alpha \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \right] \frac{1}{q} dt \right]^{\frac{1}{q}} \end{align*}
\]

\[
= \frac{\ln(b/a)}{2} \left[ \int_0^1 a^{3p(1-t)} b^{3pt} dt \right]^{\frac{1}{p}} \\
\times \left[ m_1 \left| f' \left( \frac{1}{a^{m_1}} \right) \right|^q \int_0^1 (1 - t^\alpha) dt + m_2 \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \int_0^1 t^\alpha dt \right]^{\frac{1}{q}} \]

\[
= \frac{\ln(b/a)}{2} \left[ \frac{1}{L^p(a^{3p}, b^{3p})} \right]^{\frac{1}{p}} \left[ \frac{m_1 \left| f' \left( \frac{1}{a^{m_1}} \right) \right|^q \int_0^1 \left( \frac{1}{a^{m_1}} \right)^q dt + m_2 \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \int_0^1 \left( \frac{1}{b^{m_2}} \right)^q dt} {\alpha + 1} \right]^{\frac{1}{q}} \]

This completes the proof of the theorem.

**Corollary 9.** By considering the conditions of Theorem 9, if we take \(m_1 = m_2 = 1\) in the inequality (4.3), then we get

\[
\frac{b^2 f(a) - a^2 f(b)}{2} - f_a x f(x) dx \leq \frac{\ln(b/a)}{2} \left[ \frac{1}{L^p(a^{3p}, b^{3p})} \right]^{\frac{1}{p}} \left[ \frac{m_1 \left| f' \left( \frac{1}{a^{m_1}} \right) \right|^q \int_0^1 \left( \frac{1}{a^{m_1}} \right)^q dt + m_2 \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \int_0^1 \left( \frac{1}{b^{m_2}} \right)^q dt} {\alpha + 1} \right]^{\frac{1}{q}} \]

**Corollary 10.** By considering the conditions of Theorem 9, if we take \(m_1 = m, m_2 = 1\) in the inequality (4.3) then we obtain

\[
\frac{b^2 f(a) - a^2 f(b)}{2} - f_a x f(x) dx \leq \frac{\ln(b/a)}{2} \left[ \frac{1}{L^p(a^{3p}, b^{3p})} \right]^{\frac{1}{p}} \left[ \frac{m_1 \left| f' \left( \frac{1}{a^{m_1}} \right) \right|^q \int_0^1 \left( \frac{1}{a^{m_1}} \right)^q dt + m_2 \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \int_0^1 \left( \frac{1}{b^{m_2}} \right)^q dt} {\alpha + 1} \right]^{\frac{1}{q}} \]

\[
\leq \text{in(b/a)} \frac{1}{2} \left[ \int_0^1 \left( a^{3(1-t)} b^{3t} \right)^p dt \right]^{\frac{1}{p}} + m_2 \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \int_0^1 t^\alpha dt \]

\[
= \frac{\ln(b/a)}{2} \left[ \frac{1}{L^p(a^{3p}, b^{3p})} \right]^{\frac{1}{p}} \left[ \frac{m_1 \left| f' \left( \frac{1}{a^{m_1}} \right) \right|^q \int_0^1 \left( \frac{1}{a^{m_1}} \right)^q dt + m_2 \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \int_0^1 \left( \frac{1}{b^{m_2}} \right)^q dt} {\alpha + 1} \right]^{\frac{1}{q}} \]

**Theorem 10.** Let the function \(f: \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}\) be a differentiable function and \(f^q \in L[a,b]\) for \(0 < a < b < \infty\). If \(|f|^q\) is \((\alpha, m_1, m_2)\)-GA convex on \([0, \max \left\{ \frac{1}{a^{m_1}}, \frac{1}{b^{m_2}} \right\} )\) for \([\alpha, m_1, m_2] \in (0, 1)^3\) and \(q > 1\), then the following integral inequalities hold

\[
\frac{b^2 f(a) - a^2 f(b)}{2} - f_a x f(x) dx \leq \frac{\ln(b/a)}{2} \left[ \frac{1}{L^p(a^{3p}, b^{3p})} \right]^{\frac{1}{p}} \left[ \frac{m_1 \left| f' \left( \frac{1}{a^{m_1}} \right) \right|^q \int_0^1 \left( \frac{1}{a^{m_1}} \right)^q dt + m_2 \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \int_0^1 \left( \frac{1}{b^{m_2}} \right)^q dt} {\alpha + 1} \right]^{\frac{1}{q}} \]

where \(L\) is the logarithmic mean, \(\Gamma\) is the Gamma function and \(\frac{1}{p} + \frac{1}{q} = 1\).

**Proof:** From both Lemma 1, Hölder integral inequality and the \((\alpha, m_1, m_2)\)-GA-convexity of the function \(|f|^q\) on the interval \([0, \max \left\{ \frac{1}{a^{m_1}}, \frac{1}{b^{m_2}} \right\} )\), we get

\[
\frac{b^2 f(a) - a^2 f(b)}{2} - f_a x f(x) dx \leq \frac{\ln(b/a)}{2} \left( \int_0^1 dt \right)^{\frac{1}{p}} \]

\[
\cdot \left[ \int_0^1 \left( \int_0^1 \left( \frac{1}{a^{m_1}} \right)^q dt \right)^{\frac{1}{q}} \left( \int_0^1 \left( \frac{1}{b^{m_2}} \right)^q dt \right)^{\frac{1}{q}} \right]^{\frac{1}{q}} \]

\[
\leq \frac{\ln(b/a)}{2} \left( \int_0^1 \left( \frac{1}{a^{m_1}} \right)^q dt \right)^{\frac{1}{q}} \left( \int_0^1 \left( \frac{1}{b^{m_2}} \right)^q dt \right)^{\frac{1}{q}} \]

\[
\leq \text{in(b/a)} \frac{1}{2} \left[ \int_0^1 \left( a^{3(1-t)} b^{3t} \right)^p dt \right]^{\frac{1}{p}} + m_2 \left| f' \left( \frac{1}{b^{m_2}} \right) \right|^q \int_0^1 t^\alpha dt \]

Convex function on the interval

\[ \frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx \quad (4.5) \]

where \( L \) is the logarithmic mean and \( \frac{1}{p} + \frac{1}{q} = 1. \)

**Proof.** From Lemma 1, Hölder-Işcan integral inequality and the \((a, m_1, m_2)\)-GA convexity of the function \( |f'|^q \) on the interval \( \left[ 0, \max \left\{ \frac{1}{m_1}, \frac{1}{m_2} \right\} \right] \), we obtain

\[ \frac{b^2f(a)-a^2f(b)}{2} - \int_a^b xf(x)dx \]

\[ \leq \frac{\ln b-\ln a}{2} \left[ \int_0^1 (1 - t) \left( a^{2(1-t)b^3t} \right)^p dt \right]^{\frac{1}{p}} \]

\[ + \frac{\ln b-\ln a}{2} \left[ \int_0^1 t (a^{3(1-t)b^3t})^p dt \right]^{\frac{1}{p}} \]

\[ \times \left[ \int_0^1 t \left( \left( \frac{m_1}{m_1^3} \right)^{m_1(1-t)} \left( \frac{m_2}{m_2^3} \right)^{m_2t} \right)^q dt \right]^{\frac{1}{q}} \]

\[ \leq \frac{\ln b-\ln a}{2} \left[ \int_0^1 (1 - t) a^{3p(1-t)b^{3pt}} dt \right]^{\frac{1}{p}} \]

\[ \times \left( \frac{\ln b-\ln a}{2} \left[ \int_0^1 t (a^{3(1-t)b^3t})^p dt \right]^{\frac{1}{p}} \right) \]

\[ + \frac{\ln b-\ln a}{2} \left[ \int_0^1 (1 - t)(1 - ta) \left( f'(b) \right)^q dt \right]^{\frac{1}{q}} \]

\[ + \frac{\ln b-\ln a}{2} \left[ \int_0^1 (1 - t) \left( f'(b) \right)^q dt \right]^{\frac{1}{q}} \]

\[ + \frac{\ln b-\ln a}{2} \left[ \int_0^1 (1 - t)(1 - ta) \left( f'(b) \right)^q dt \right]^{\frac{1}{q}} \]

\[ + \frac{\ln b-\ln a}{2} \left[ \int_0^1 (1 - t)(1 - ta) \left( f'(b) \right)^q dt \right]^{\frac{1}{q}} \]

\[ + \frac{\ln b-\ln a}{2} \left[ \int_0^1 (1 - t)(1 - ta) \left( f'(b) \right)^q dt \right]^{\frac{1}{q}} \]

\[ + \frac{\ln b-\ln a}{2} \left[ \int_0^1 (1 - t)(1 - ta) \left( f'(b) \right)^q dt \right]^{\frac{1}{q}} \]
Theorem 11, if we take $m_1 = m_2 = 1$ in the inequality (4.5), then we get

$$\frac{\ln b - \ln a}{2} \left[ \frac{\left( a^{3p}, b^{3p} \right) - a^{3p}}{3(\ln b - \ln a)} \right] \frac{1}{p}
\begin{align*}
&= \frac{\ln b - \ln a}{2} \left[ \frac{\left( a^{3p}, b^{3p} \right) - a^{3p}}{3(\ln b - \ln a)} \right] \frac{1}{p} \\
&\quad + \left[ f' \left( \frac{1}{m_1} \right) \right]^q \left( \frac{a(a+3)m_1}{2(2a+3a+2)} \right) + f' \left( \frac{1}{m_2} \right) \left( \frac{m_2}{a+2} \right) \frac{1}{q}. \\
&\text{This completes the proof of the theorem.}
\end{align*}$$

5. CONCLUSION

New Hermite-Hadamard type integral inequalities can be obtained by using $(\alpha, m_1, m_2)$-GA convexity and different type identities.

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