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Spectral Analysis of Non-selfadjoint Second Order Difference Equation with Operator Coefficient

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Abstract

In this paper, we consider the discrete Sturm-Liouville operator generated by second order difference equation with non-selfadjoint operator coefficient. This operator is the discrete analogue of the Sturm-Liouville differential operator generated by Sturm-Liouville operator equation which has been studied in detail. We find the Jost solution of this operator and examine its asymptotic and analytical properties. Then, we find the continuous spectrum, the point spectrum and the set of spectral singularities of this discrete operator. We finally prove that this operator has a finite number of eigenvalues and spectral singularities under a specific condition.

Keywords: Sturm-Liouville’s operator equation, Non-selfadjoint operators, Discrete operators, Continuous spectrum, Operator coefficients.

1. INTRODUCTION

Difference equations are very important for modelling certain problems in physics, biology, economics, engineering, control theory etc. Spectral analysis of certain difference equations gives us useful information about these problems.

Let us give some literature on the spectral analysis of non-selfadjoint operators and the concept of spectral singularities. Spectral analysis of non-selfadjoint Sturm-Liouville operator has begun and the spectral singularities was discovered by Naimark [1-2]. Spectral singularities of differential operators [3-4] and certain classes of abstract operators [5] are studied.

Recently, non-Hermitian Hamiltonians and complex extension of quantum mechanics have been studied extensively (see review papers [6-7]). Moreover, the spectral singularities are identified for some concrete complex scattering potentials and some physical interpretations are suggested [8-9]. In [9], the authors identify the spectral singularities of complex scattering potentials with the real energies at which the reflection and transmission coefficients tend to
infinity, i.e., they correspond to resonances having a zero width.

Spectral analysis of selfadjoint difference equations has been studied by many authors (see [10-11] for review and references). Further, spectral analysis of the selfadjoint differential and difference equations with matrix coefficients has been investigated [12-16]. In [17-19], the authors investigated the difference equation

\[ a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}, \]  

where \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) are complex sequences such that an \(a_n \neq 0\) for all \(n \in \mathbb{N}\) and the condition

\[ \sum_{n=1}^{\infty} n (|1 - a_n| + |b_n|) < \infty \]  

holds. We can refer to Equation (1) as the Sturm-Liouville difference equation since it can be rewritten

\[ \Delta(a_{n-1} \Delta y_{n-1}) + q_n y_n = \lambda y_n, \quad n \in \mathbb{N}, \]

where \(q_n = a_{n-1} + a_n + b_n\) and \(\Delta\) denotes the forward difference operator.

In [20-21], the authors considered the case \(n \in \mathbb{Z}\) with the analogous condition to (2). Further, the spectral analysis of the Sturm-Liouville difference equation with finite dimensional non-Hermitian matrix coefficients has been done [22].

Although there are many studies on spectral properties of the Sturm-Liouville difference equation with scalar or finite dimensional matrix coefficients, there isn’t any study when the coefficients are infinite dimensional operators. In scalar or matrix coefficient cases, the discrete spectrum and spectral singularities are obtained as zeros of Jost function by using the results about analytic scalar functions. The infinite dimensional case requires a different treatment and new methods since the Jost function is an operator function on the contrary to finite dimensional case. The new method is due to Keldysh [23] which gives the fundamental tools to examine the singular points of analytic operator functions.

We consider the following difference operator defined in the Hilbert space \(H_1 := l_2(\mathbb{N}, H)\) of vector sequences \(y = (y_n)_{n \in \mathbb{N}} \in \mathbb{H}\) such that

\[ \sum_{n=1}^{\infty} ||y_n||_H^2 < \infty, \]

where \(H\) is a separable Hilbert space \((dimH \leq \infty)\).

Let us denote the difference operator \(L\) defined in \(H_1\):

\[ l(y)_n := A_{n-1} y_{n-1} + B_n y_n + A_n y_{n+1}, \quad n \in \mathbb{N}, \]

\[ y_0 = 0, \]

where \(A_n \ (n \in \mathbb{N} \cup \{0\})\) and \(B_n \ (n \in \mathbb{N})\) are non-selfadjoint, \(A_n - I \ (n \in \mathbb{N} \cup \{0\})\) and \(B_n \ (n \in \mathbb{N})\) are completely continuous operators in \(H\) such that \(A_n\) is invertible for \(n \in \mathbb{N} \cup \{0\}\).

In this paper, we investigate the spectral properties of the non-selfadjoint difference operator \(L\) which is generated by the Sturm-Liouville difference equation with non-selfadjoint operator coefficients. In particular, we find the Jost solution, continuous spectrum, discrete spectrum and spectral singularities of \(L\). Finally, we prove the finiteness of eigenvalues and spectral singularities.

2. THE JOST SOLUTION AND CONTINUOUS SPECTRUM OF \(L\)

Let us consider the eigenvalue equation of \(L\)

\[ A_{n-1} y_{n-1} + B_n y_n + A_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}, \]

where \(\lambda\) is the spectral parameter. Equivalently, we might consider the corresponding operator equation

\[ A_{n-1} Y_{n-1} + B_n Y_n + A_n Y_{n+1} = \lambda Y_n, \quad n \in \mathbb{N}, \]

where \((Y_n)\) is an operator sequence i.e. \(Y_n\) is an operator in \(H\) for each \(n \in \mathbb{N}\).

**Assumption 1.** We assume that the coefficients of Equation (5) satisfy

\[ \sum_{n=1}^{\infty} (||I - A_n|| + ||B_n||) < \infty. \]
Definition 1. Let \( E(z) := E_n(z) \) \((n \in \mathbb{N} \cup \{0\})\) denote the operator solution of the equation
\[
A_{n-1} Y_{n-1} + B_n Y_n + A_n Y_{n+1} = (z + z^{-1}) Y_n, \quad n \in \mathbb{N},
\]
satisfying the condition
\[
\lim_{n \to \infty} Y_n(z) z^{-n} = I,
\]
for \( z \in D_0 := \{z \in \mathbb{C}: |z| = 1\} \). \( E(z) \) is called the Jost solution of Equation (5).

Remark 1. The remaining results of this section will be given without proofs since they are similar to the matrix coefficient case which have been obtained in [12].

Assumption 2. Let us assume
\[
\sum_{n=1}^{\infty} n(\|I - A_n\| + \|B_n\|) < \infty.
\]

Theorem 1. The Jost solution can be represented
\[
E_n(z) = T_n z^n \left[ I + \sum_{m=1}^{\infty} K_{n,m} z^m \right], \quad n \in \mathbb{N} \cup \{0\}, \quad (6)
\]
where
\[
T_n = \prod_{p=n}^{\infty} A_p^{-1},
\]
\[
K_{n,1} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p,
\]
\[
K_{n,2} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p,1} + \sum_{p=n+1}^{\infty} T_p^{-1} (I - A_p^2) T_p,
\]
\[
K_{n,m+2} = \sum_{p=n+1}^{\infty} T_p^{-1} (I - A_p^2) T_p K_{p+1,m} - \sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p,m+1} + K_{n+1,m},
\]
where \( n \in \mathbb{N} \cup \{0\} \), \( m \in \mathbb{N} \). Further,
\[
\|K_{n,m}\| \leq c \sum_{p=n+1}^{\infty} \left( \|I - A_p\| + \|B_p\| \right)
\]
holds where \( c > 0 \) is a constant. Moreover, \( E_n(z) \) \((n \in \mathbb{N} \cup \{0\})\) has an analytic continuation from \( D_0 \) to \( D_1 := \{z \in \mathbb{C}: |z| < 1\} \) \(\backslash\{0\}\).

Theorem 2. The Jost solution satisfies the asymptotic relation
\[
E_n(z) = z^n [I + o(1)], \quad n \to \infty,
\]
for \( z \in D := \{z \in \mathbb{C}: |z| \leq 1\} \backslash\{0\} \).

Theorem 3. The continuous spectrum of \( L \) is \( \sigma_c(L) = [-2, 2] \).

Proof. Let \( L_0 \) and \( L_1 \) denote the operators defined in \( H_1 \)
\[
L_0(y)_n = y_{n-1} + y_{n+1}, \quad n \in \mathbb{N},
\]
\[
L_1(y)_n = (A_{n-1} - I)y_{n-1} + B_n y_n + (A_n - I)y_{n+1}, \quad n \in \mathbb{N},
\]
with the boundary condition
\[
y_0 = 0,
\]
respectively. It easily follows \( L_0 = L_0^* \) and also \( \sigma_c(L_0) = [-2, 2] \) (see [24]).

It is well known that \( L_1 \) is a compact operator iff \( L_1 \) is bounded and the set
\[
R = \{L_1 y: \|y\|_1 \leq 1\}
\]
is compact in \( H_1 \). It is obvious that \( L_1 \) is bounded. Moreover, if we use the compactness criteria in \( l_p \) spaces (see [25], p. 167), we obtain the compactness of \( R \). Indeed, let \( y \in H_1 \) such that \( \|y\|_1 \leq 1 \). Then, Assumption 2 implies that for \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for \( n \geq n_0 \)
\[
\sum_{i=n+1}^{\infty} (\|A_i - I\| + \|B_i\|) < \frac{\varepsilon}{c}
\]
holds and also
\[
\sum_{i=n+1}^{\infty} \|L_1 y\|_1^2 = \sum_{i=n+1}^{\infty} (A_{i-1} - I) y_{i-1} + B_i y_i + (A_i - I) y_{i+1} \|1_y\|_1^2 \\
\leq \sum_{i=n+1}^{\infty} \|A_{i-1} - I\|^2 \|y_{i-1}\|_1^2 + \|B_i\|^2 \|y_i\|_1^2 + \|A_i - I\|^2 \|y_{i+1}\|_1^2 \\
\leq \|y\|_1^2 (\sum_{i=n+1}^{\infty} \|A_{i-1} - I\|^2 + \|B_i\|^2 + \|A_i - I\|^2) \\
\leq \sum_{i=n+1}^{\infty} \|A_i - I\|^2 + \|B_i\|^2 \\
\leq \sum_{i=n+1}^{\infty} C_1 \|A_i - I\| + C_2 \|B_i\| \\
\leq \sum_{i=n+1}^{\infty} C (\|A_i - I\| + \|B_i\|) < \varepsilon,
\]
where
\[ C_1 = \frac{1}{2} \sup_{i \in \mathbb{N}} \|A_i - I\|, \quad C_2 = \sup_{i \in \mathbb{N}} \|B_i\|, \quad C = C_1 + C_2. \]

Therefore, we proved that \( L_1 \) is a compact operator in \( H_1 \). Weyl’s theorem of compact perturbation [26] implies
\[ \sigma_c(L) = \sigma_c(L_0) = [-2, 2]. \]

### 3. EIGENVALUES AND SPECTRAL SINGULARITIES OF \( L \)

It is easy to show that the discrete spectrum and the set of spectral singularities of \( L \) are
\[
\sigma_d(L) = \{ \lambda: \lambda = z + z^{-1}, z \in D_1, E_0(z) \text{ is not invertible} \},
\]
\[
\sigma_{ss}(L) = \{ \lambda: \lambda = z + z^{-1}, z \in D_0, E_0(z) \text{ is not invertible} \},
\]
respectively. \( E_0(z) \) is called the Jost function of \( L \). Note that, this function is an operator function on the contrary to finite dimensional case. Hence, the methods need to be changed in our case. We will use Keldysh [23] in order to analyze the singular points of \( E_0(z) \). Let us define the sets
\[ M_1 = \{ z \epsilon D_1: E_0(z) \text{ is not invertible} \}, \]
\[ M_2 = \{ z \epsilon D_0: E_0(z) \text{ is not invertible} \}. \]

Then,
\[
\sigma_d(L) = \{ \lambda: \lambda = z + z^{-1}, z \epsilon M_1 \},
\]
\[
\sigma_{ss}(L) = \{ \lambda: \lambda = z + z^{-1}, z \epsilon M_2 \}.
\]

From the representation (6) we have
\[ E_0(z) = T_0 \left[ I + \sum_{m=1}^{\infty} K_{0,m} z^m \right], \]
where
\[ T_0 = \prod_{p=0}^{\infty} A_p^{-1} \]
is invertible. This implies \( E_0(z) \) is invertible iff \( F(z) := I + \sum_{m=1}^{\infty} K_{0,m} z^m \)
is invertible. Hence
\[ M_1 = \{ z \epsilon D_1: F(z) \text{ is not invertible} \}. \]

From Theorem 1 we have
\[
K_{0,1} = -\sum_{p=1}^{\infty} T_p^{-1} B_p T_p, \quad K_{0,2} = -\sum_{p=1}^{\infty} T_p^{-1} B_p T_p K_{p,1} + \sum_{p=1}^{\infty} T_p^{-1} (I - A_p^2) T_p,
\]
\[
K_{0,m+2} = \sum_{p=1}^{\infty} T_p^{-1} (I - A_p^2) T_p K_{p+1,m} - \sum_{p=1}^{\infty} T_p^{-1} B_p T_p K_{p+1,m} + K_{1,m},
\]
where \( m \in \mathbb{N} \). These equations together with the conditions that \( A_n - I (n \in \mathbb{N} \cup \{0\}) \) and \( B_n (n \in \mathbb{N}) \) are completely continuous operators imply that \( K_{0,m} \) is completely continuous for every \( m \in \mathbb{N} \). As a result,
\[ K(z) := \sum_{m=1}^{\infty} K_{0,m} z^m \]
is a completely continuous operator for every \( z \in D_1 \). Moreover, since \( E_0(z) \) is analytic on \( D_1 \) (see Theorem 1) \( F(z) \) is also analytic on \( D_1 \). Hence, we can use [23] to find the singular points of the operator valued function \( F(z) \) on \( D_1 \) and these singular points give us the eigenvalues.

**Definition 2.** [23] Let \( A \) and \( R \) be operators such that
\[
(I + R)(I - A) = I
\]
holds. Then, \( R \) is called the resolvent of \( A \).

**Theorem 4.** Let \( R(z) \) denote the resolvent of \( -K(z) \). Then,
\[ M_1 = \{ z \epsilon D_1: z \text{ is a pole of } I + R(z) \}. \]

**Proof.** Since \( R(z) \) is the resolvent of \( -K(z) \), we have
\[
I + R(z) = (I + K(z))^{-1} = (F(z))^{-1}.
\]
Since \( M_1 \neq D_1 \) there exists \( z \epsilon D_1 \) such that \( I + R(z) \) exists. Therefore, \( I + R(z) \) exists on \( D_1 \) except for a set of isolated points and \( I + R(z) \) is
a meromorphic function of \( z \) on \( D_1 \) [23]. These isolated points are clearly the eigenvalues of \( L \). Hence we have

\[
(F(z))^{-1} = \frac{u(z)}{v(z)}, \quad z \in D_1, \tag{7}
\]

where \( U(z) \) is an operator function and \( v(z) \) is a scalar function which are both analytic on \( D_1 \).

Since

\[
M_1 = \{ z \in D_1 : F(z) \text{ is not invertible} \},
\]

it follows from (7) that

\[
M_1 = \{ z \in D_1 : z \text{ is a pole of } I + R(z) \} = \{ z \in D_1 : v(z) = 0 \}. \tag{8}
\]

Corollary 1. \( \sigma_d(L) = \{ \lambda : \lambda = z + z^{-1}, z \in D_1, v(z) = 0 \} \).

Proof. The proof is obvious from Theorem 4 since \( \sigma_d(L) = \{ \lambda : \lambda = z + z^{-1}, z \in M_1 \} \).

Theorem 5. \( \sigma_d(L) \) is bounded and countable. Moreover, its limit points can only lie in the circle \( |z| \leq 2 \).

Proof. From (6) it follows

\[
E_0(z) = T_0[I + \sum_{m=1}^{\infty} K_{0,m} z^m]
\]

and also

\[
E_0(z) = T_0I + o(1), \quad |z| \to 0,
\]

which implies \( E_0(z) \) is invertible for sufficiently small \( z \) and hence \( \lambda = z + z^{-1} \) is not an eigenvalue for \( |z| \to 0 \). Thus, \( \sigma_d(L) \) is bounded. Since \( v(z) \) is analytic on \( D_1 \), its zeros are isolated. From Corollary 5, it follows \( \sigma_d(L) \) is countable. Further, the limit points (if exist) of the zeros of \( v(z) \) lie on the boundary of \( D_1 \) i.e. on \( D_0 \) [27].

If \( z \) is a limit point of the set

\[
M_1 = \{ z \in D_1 : v(z) = 0 \},
\]

then \( |z| = 1 \) and \( |\lambda| = |z + z^{-1}| \leq 2 \). Hence, the limit points (if exist) lie in the circle \( |z| \leq 2 \).

Theorem 6. Let \( R(z) \) denote the resolvent of \( -K(z) \). Then,

\[
M_2 = \{ z \in D_0 : z \text{ is a pole of } I + R(z) \} = \{ z \in D_0 : v(z) = 0 \}. \tag{9}
\]

Proof. From (7) it follows

\[
F(z)U(z) = v(z)I,
\]

which implies \( U(z) \) is invertible whenever \( v(z) \neq 0 \). Moreover, if \( z \in D_1 \) such that \( v(z) \neq 0 \) then

\[
F(z) = v(z)(U(z))^{-1}.
\]

Since \( F(z) \) is continuous on \( D \), we can extend this representation continuously to \( D \);

\[
(F(z))^{-1} = \frac{u(z)}{v(z)}, \quad z \in D. \tag{10}
\]

Recall that

\[
M_2 = \{ z \in D_0 : F(z) \text{ is not invertible} \}.
\]

The representation (10) implies that

\[
M_2 = \{ z \in D_0 : v(z) = 0 \}. \tag{11}
\]

Theorem 7. The set of spectral singularities \( \sigma_{ss}(L) \) of \( L \) is compact and has zero Lebesgue measure.

Proof. It is well known that \( \sigma_{ss}(L) \subset \sigma_c(L) = [-2,2] \). This gives us the boundedness of \( \sigma_{ss}(L) \). We only have to show \( \sigma_{ss}(L) \) or equivalently \( M_2 \) is closed for the compactness. Let \( \{ \lambda_n \} \subset M_2 \) such that \( \lambda_n \to \lambda \). \( \{ \lambda_n \} \subset M_2 \) implies \( \lambda_n \in D_0 \) and \( v(\lambda_n) = 0 \) for every \( n \in \mathbb{N} \). Since \( D_0 \) is closed \( \lambda_n \to \lambda \) implies \( \lambda \in D_0 \). Moreover, since \( v(z) \) is continuous on \( D \), \( \lambda_n \to \lambda \) implies \( v(\lambda_n) \to v(\lambda) \) and hence \( v(\lambda) = 0, \lambda \in M_2 \) as required. Finally, since

\[
M_2 = \{ z \in D_0 : v(z) = 0 \},
\]

it follows from Privalov’s Theorem [27] that \( M_2 \) has zero Lebesgue measure.
**Assumption 3.** Let us assume
\[
\sum_{n=0}^{\infty} e^{\varepsilon n} (\|I - A_n\| + \|B_n\|) < \infty, \quad \varepsilon > 0.
\]

**Theorem 8.** L has a finite number of eigenvalues and spectral singularities.

**Proof.** Recall that
\[
\|K_{n,m}\| \leq c \sum_{p=0}^{\infty} \left( \|I - A_p\| + \|B_p\| \right),
\]
where c is a constant. This implies together with the Assumption 3 that
\[
\|K_{0,m}\| \leq C e^{-\frac{\varepsilon m}{6}}, \quad m \in \mathbb{N},
\]
where C > 0. The series
\[
\sum_{m=0}^{\infty} K_{0,m} z^m, \quad \sum_{m=0}^{\infty} m K_{0,m} z^{m-1}
\]
are uniformly convergent iff \(\ln|z| < \frac{e^\varepsilon}{6}\). Hence, \(E_n(z) (n \in \mathbb{N} \cup \{0\})\) has an analytic continuation to \(D_2^2 = \{ z \in \mathbb{C} : |z| < e^{\frac{e^\varepsilon}{6}} \}\). Thus, we can write
\[
(F(z))^{-1} = \frac{v(z)}{v(z)}, \quad z \in D_2.
\]
Let us suppose that \(M_1\) and \(M_2\) are not finite. Since \(M_1\) and \(M_2\) are bounded (see Theorem 5 and 7), they have limit points by Bolzano-Weierstrass Theorem. Since \(v(z)\) is analytic on \(D_2\), the limit points of its zeros must lie on the boundary of the domain \(D_2\) [27]. This gives a contradiction since \(e^{\frac{e^\varepsilon}{6}} > 1\). Therefore, \(M_1\) and \(M_2\) must be finite.

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