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Authors: Öznur Özkan Kılıç
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Coefficient Inequalities for Janowski Type Close-to-Convex Functions Associated with Ruscheweyh Derivative Operator

Öznur Özkan Kılıç

ABSTRACT

The aim of this paper is to introduce a new subclasses of the Janowski type close-to-convex functions defined by Ruscheweyh derivative operator and obtain coefficient bounds belonging to this class.

Keywords: Univalent Function, Subordination, Close-to-Convex Function, Ruscheweyh Derivative Operator

1. INTRODUCTION

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk

\[\Delta = \{ z \in \mathbb{C} : |z| < 1 \}.\]

Let \( \mathcal{S} \) denote the subclasses of \( \mathcal{A} \) which are univalent in \( \Delta \).

An analytic function \( f \) is subordinate to an analytic function \( F \), written as \( f \prec F \) or \( f(z) \prec F(z) \), if there exists a Schwarz function \( \omega : \Delta \to \Delta \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) satisfying \( f(z) = F(\omega(z)) \). In particular, if \( F \) is univalent in \( \Delta \), we have the following equivalence:

\( f(z) \prec F(z) \iff [f(0) = F(0) \land f(\Delta) = F(\Delta)]. \)

The Hadamard product or convolution of two functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A} \) and

\( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} \), denoted by \( f \ast g \), is defined by

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n
\]

for \( z \in \Delta \).

In 1975, Ruscheweyh [5] introduced a linear operator \( \mathcal{D}^\delta : \mathcal{A} \to \mathcal{A} \) defined by

\[
\mathcal{D}^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} \ast f(z)
\]

\[
= z + \sum_{n=2}^{\infty} \varphi_n(\delta) a_n z^n
\]

with

\[
\varphi_n(\delta) = \frac{(\delta + 1)_{n-1}}{(n-1)!}
\]

* Corresponding Author

1 Baskent University, Statistics and Computer Science Program, Ankara, Turkey ORCID: 0000-0003-4209-9320
for \( \delta > -1 \) and \( (a)_n \) is Pochhammer symbol defined by

\[
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} \quad \text{if } n = 0
\]

\[
= \begin{cases} 
1 & \text{if } n = 0 \\
(a(a + 1) \cdots (a + n - 1) & \text{if } n \in \mathbb{N}
\end{cases}
\]

for \( a \in \mathbb{C} \) and \( \mathbb{N} = \{1,2,3,\ldots\} \).

Notice that

\[
D^0 f(z) = f(z),
\]

\[
D^1 f(z) = z f'(z)
\]

and

\[
D^n f(z) = \frac{z(z^{m-1} f(z))^m}{m!}
\]

\[
= z + \sum_{n=2}^{\infty} \frac{\Gamma(n + m)}{\Gamma(m + 1)(n-1)!} a_n z^n
\]

for all \( \delta = m \in \mathbb{N}_0 = \{0,1,2,\ldots\} \).

In geometric function theory, various subclasses defined by Ruscheweyh derivative operator were studied.

Let \( \mathcal{S}^* \) and \( \mathcal{C} \) be the usual subclasses of functions which members are univalent, starlike and convex in \( \Delta \), respectively. We also denote \( \mathcal{S}^*(\alpha) \) and \( \mathcal{C}(\alpha) \) the class of starlike functions of order \( \alpha \) and the class of convex functions of order \( \alpha \), for \( 0 \leq \alpha < 1 \), respectively. Note that \( \mathcal{S}^* = \mathcal{S}^*(0) \) and \( \mathcal{C} = \mathcal{C}(0) \).

In 1973, Janowski [2] introduced the classes by \( \mathcal{S}^*(A,B) \) and \( \mathcal{C}(A,B) \)

\[
\mathcal{S}^*(A,B) = \left\{ f \in \mathcal{A} : \frac{zf''(z)}{f'(z)} < \frac{1 + Az}{1 + Bz}, g \in \mathcal{S}^* \right\}
\]

and

\[
\mathcal{C}(A,B) = \left\{ f \in \mathcal{A} : 1 + \frac{zf'''(z)}{f''(z)} < \frac{1 + Az}{1 + Bz} \right\}
\]

for \(-1 \leq B < A \leq 1\), \( z \in \Delta \). Note that \( \mathcal{S}^*(\alpha) = \mathcal{S}^*(1 - 2\alpha, -1) \), \( \mathcal{S}^* = \mathcal{S}^*(1, -1) \) and \( \mathcal{C}(\alpha) = \mathcal{C}(1 - 2\alpha, -1) \), \( \mathcal{C} = \mathcal{C}(1, -1) \).

A function \( f \in \mathcal{A} \) is said to be close-to-star if and only if there exists \( g \in \mathcal{S}^* \) such that \( \Re\{f(z)/g(z)\} > 0 \) for all \( z \in \Delta \). Also, a function \( f \in \mathcal{A} \) is said to be close-to-convex if and only if there exists \( g \in \mathcal{C} \) such that \( \Re\{f'(z)/g'(z)\} > 0 \) for all \( z \in \Delta \). The classes of close-to-star and close-to-convex functions denote by \( \mathcal{CS}^*(\gamma) \) and \( \mathcal{CC}(\gamma) \), respectively. The class of close-to-star functions was introduced by Reade in [4] and the class of close-to-convex functions was introduced by Kaplan in [3]. Similarly, we denote by \( \mathcal{CS}^*(\gamma) \) and \( \mathcal{CC}(\gamma) \) the classes of close-to-star functions of order \( \gamma \) and close-to-convex functions of order \( \gamma \), for \( 0 \leq \gamma < 1 \), respectively. Note that \( \mathcal{CS}^* = \mathcal{CS}^*(0) \) and \( \mathcal{CC} = \mathcal{CC}(0) \).

The class of Janowski type close-to-starlike functions in \( \Delta \), denoted by \( \mathcal{CS}^*(A,B) \), is defined by

\[
\mathcal{CS}^*(A,B) = \left\{ f \in \mathcal{A} : \frac{f(z)}{g(z)} < \frac{1 + Az}{1 + Bz}, g \in \mathcal{S}^* \right\}
\]

for \(-1 \leq B < A \leq 1\), \( z \in \Delta \). Similarly, the class of Janowski type close-to-convex functions in \( \Delta \), denoted by \( \mathcal{CC}(A,B) \), is defined by

\[
\mathcal{CC}(A,B) = \left\{ f \in \mathcal{A} : \frac{f'(z)}{g'(z)} < \frac{1 + Az}{1 + Bz}, g \in \mathcal{C} \right\}
\]

for \(-1 \leq B < A \leq 1\), \( z \in \Delta \). The classes are introduced by Reade [4] in 1955.

**Definition 1.1.** The class of Janowski type functions defined by Ruscheweyh derivative operator in \( \Delta \), denoted by \( \mathcal{J}_\delta(\delta,\beta,A,B) \), is defined by

\[
\mathcal{J}_\delta(\delta,\beta,A,B) = \left\{ f \in \mathcal{A} : \frac{D^\delta f(z)}{D^\beta g(z)} < \frac{1 + Az}{1 + Bz} \right\}
\]

for \( \delta,\beta > -1, -1 \leq B < A \leq 1, z \in \Delta \).

We need the following lemma to obtain our results.

**Lemma 1.2.** [1] If the function \( p(z) \) of the form
\[ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \]
is analytic in \( \Delta \) and
\[ p(z) < \frac{1 + Az}{1 + Bz} \]
then \( |p_n| \leq A - B \) for \( n \in \mathbb{N}, -1 \leq B < A \leq 1 \).

2. MAIN RESULTS AND THEIR CONSEQUENCES

We begin by finding the estimates on the coefficient \( |a_n| \) for functions in the class \( J_R(\delta, \beta, A, B) \).

**Theorem 2.1.** If the function \( f(z) \in \mathcal{A} \) be in the class \( J_R(\delta, \beta, A, B) \), then

\[ |a_n| \leq \frac{n \varphi_n(\beta) + (A - B) \sum_{m=1}^{n-1} m \varphi_m(\beta)}{\varphi_n(\delta)}. \quad (2.1) \]

**Proof.** Let \( f(z) \in J_R(\delta, \beta, A, B) \). Then, there are analytic functions \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \), \( \omega \) is a Schwarz function and

\[ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \] \as in Lemma 1.2 such that

\[ \frac{D^\delta f(z)}{D^\beta g(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} = p(z) \quad (2.2) \]

for \( z \in \Delta \). Then (2.2) can be written as

\[ D^\delta f(z) = p(z).D^\beta g(z) \]
or

\[ z + \sum_{n=2}^{\infty} \varphi_n(\delta) a_n z^n \]

\[ = z + \sum_{n=2}^{\infty} \sum_{m=1}^{n} \varphi_{n-m+1}(\beta) b_{n-m+1} p_{m-1} \]

Equating the coefficients of like powers of \( z \), we obtain

\[ \varphi_2(\delta) a_2 = \varphi_2(\beta) b_2 + p_1, \]
\[ \varphi_3(\delta) a_3 = \varphi_2(\beta) b_2 p_1 + \varphi_3(\beta) b_3 + p_2, \]
and

\[ \varphi_n(\delta) a_n = \varphi_n(\beta) b_n + \varphi_{n-1}(\beta) b_{n-1} p_1 + \varphi_{n-2}(\beta) b_{n-2} p_{2+\cdots+p_{n-1}}. \]

By using Lemma 1.2 and \( g \in \mathcal{S}^* \), we get

\[ \varphi_n(\delta)|a_n| \leq n \varphi_n(\beta) + (A - B) \sum_{m=1}^{n-1} m \varphi_m(\beta) \]
and this inequality is equivalent to (2.1).

**Corollary 2.2.** If the function \( f(z) \in \mathcal{A} \) be in the class \( \mathcal{CS}^*(A, B) \), then

\[ |a_n| \leq n + \frac{(A - B)(n - 1)n}{2}. \]

**Proof.** In Theorem 2.1, we take \( \delta = 0, \beta = 0 \).

**Corollary 2.3.** If the function \( f(z) \in \mathcal{A} \) be in the class \( \mathcal{CS}^*(\gamma) \), then

\[ |a_n| \leq n + (1 - \gamma)(n - 1)n. \]

**Proof.** In Theorem 2.1, we take \( \delta = 0, \beta = 0, A = 1 - 2\gamma, B = -1 \).

**Corollary 2.4.** If the function \( f(z) \in \mathcal{A} \) be in the class \( \mathcal{CS}^* \), then

\[ |a_n| \leq n^2. \]

**Proof.** In Theorem 2.1, we take \( \delta = 0, \beta = 0, A = 1, B = -1 \).

**Corollary 2.5.** If the function \( f(z) \in \mathcal{A} \) be in the class \( \mathcal{CC}(A, B) \), then

\[ |a_n| \leq 1 + \frac{(A - B)(n - 1)}{2}. \]

**Proof.** In Theorem 2.1, we take \( \delta = 1, \beta = 0 \).

**Corollary 2.6.** If the function \( f(z) \in \mathcal{A} \) be in the class \( \mathcal{CC}(\gamma) \), then

\[ |a_n| \leq 1 + (1 - \gamma)(n - 1). \]

**Proof.** In Theorem 2.1, we take \( \delta = 1, \beta = 0, A = 1 - 2\gamma, B = -1 \).

**Corollary 2.7.** If the function \( f(z) \in \mathcal{A} \) be in the class \( \mathcal{CC} \), then

\[ |a_n| \leq n. \]
Proof. In Theorem 2.1, we take \( \delta = 1, \ \beta = 0, \)
\( A = 1, B = -1. \)

We note that Results in Corollary 2.4 and Corollary 2.7 were proved by Reade in 1955.

(See [4])

REFERENCES


