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Long-time Behaviour of Solutions to Inverse Problem for Higher-order Parabolic Equation

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ABSTRACT

We consider an inverse problem for the fourth-order parabolic equation. Long-time behavior of the solution for the higher-order nonlinear inverse problem is established. Additional condition is given in the form of integral overdetermination.

Keywords: long-time behavior, integral overdetermination, higher-order parabolic equation, inverse problem

1. INTRODUCTION

In this study it is considered the following problem for the fourth-order parabolic equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \Delta^2 u - \sum_{i=1}^{n} b_i u_{x_i x_i} + |u|^p u &= f(t)w(x), x \in \Omega, t > 0 \\
u &= \Delta u = 0, x \in \partial \Omega, t \geq 0 \\
u(x,0) &= u_0(x), x \in \Omega \\
\int_{\Omega} u(x,t)w(x)dx &= \varphi(t)
\end{align*}
\]

where $\Omega \subset \mathbb{R}^n$ is the bounded region which has a smooth boundary $\partial \Omega$ and $p > 0$. $u_0, w, \varphi$ are given functions and they satisfy the following conditions:

\[
\begin{align*}
w &\in H_0^2(\Omega) \cap L^{p+2}(\Omega), \int_{\Omega} w^2dx = 1 \\
B_0 &= \max_{x \in \Omega} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}, x \in \Omega, b_i \in C(\overline{\Omega}).
\end{align*}
\]

The inverse problem consists of finding a pair of functions \{u(x,t), f(t)\} satisfying (1)-(4) when

\[
u_0 \in H_0^1(\Omega) \cap L^{p+2}(\Omega), \int_{\Omega} u_0wdx = \varphi(0). \quad (A3)
\]

Condition (4) is the overdetermination condition for the inverse problem and is given sometimes point-wise or integral form. Here is the integral form. Physically it means measurements of the temperature $u(x,t)$ by a device averaging over the domain $\Omega$ [4].

Inverse problems are known as ill-posed problems in Hadamard’s sense and have many application areas in physics and engineering. For example inverse scattering problems in quantum physics, inverse problems in geophysics[2].

There are some papers devoted to the study of existence and uniqueness of solutions of inverse problems for various parabolic type equations with unknown source functions [1,3,5]. In [6,7,8,9,10]
authors studied global nonexistence and blow-up solution to fourth-order equations.

Here, we used the following notations:

\[(u,v) = \int_{\Omega} uv dx, \|u(t)\| = \|u(t)\|_{L^2(\Omega)},\]

\[\|u(t)\|_p = \|u(t)\|_{L^p(\Omega)}.\]

Cauchy inequality with epsilon and Young inequality for \(a, b > 0\) are as follows;

\[ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2, \quad ab \leq \beta a^p + C(p, \beta)b^q \quad (5)\]

with \(1/p + 1/q = 1, C(p, \beta) = 1/q(\beta p)^{q/p}.\)

\[\lambda_1 \|u(t)\|^2 \leq \|\Delta u(t)\|^2, \quad u \in H^2_0(\Omega) \quad (6)\]

where \(\lambda_1\) is the least eigenvalue of the following eigenvalue problem

\[\lambda u = \Delta^2 u, \quad x \in \Omega, \quad u = \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega} = 0, \quad x \in \partial \Omega \quad [11].\]

**Definition 1** The pair of functions \(\{u(x, t), f(t)\}\) is called the weak solution of the inverse problem (1)-(4) if

\[u \in L^2(0, T; H^4(\Omega) \cap H^2_0(\Omega)) \cap L^{p+2}(\Omega), \quad f \in L^2(0, T)\]

satisfying the identity

\[(u_t, \psi) + (\Delta \psi, \Delta u) - (\psi_t \sum_{i=1}^n b_i u_{x_i x_i}) + (\psi, |u|^p u) = f(t)(\psi, w) \quad (7)\]

where \(\psi \in C_0^\infty(\Omega).\)

Multiplying both sides of (1) by \(w\) and integrating the resulting equation over \(\Omega\) leads to the following relation with the conditions (3), (4) and (A1)

\[f(t) = \varphi'(t) + (\Delta w, \Delta u) - (w, \sum_{i=1}^n b_i u_{x_i x_i}) + (w, |u|^p u). \quad (8)\]

Substituting (8) into the equation (1), the problem (1)-(3) yields a direct problem given by [1].

2. A PRIORI ESTIMATES

**Theorem 1.** Suppose that the conditions (A2) and (A3) are satisfied and assume that \(\varphi\) and \(\varphi'\) are continuous functions defined on \([0, \infty)\) which tends to zero as \(t \to \infty\). Then

\[
\lim_{t \to \infty} \left(\|\Delta u\|^2 + \frac{1}{p+2}\|u\|_{p+2}^{p+2}\right) = 0 \quad (9)
\]

with a constant \(1 > \frac{(\lambda_1 + \lambda_2)B_0^2}{4\lambda_1}\), where \(\lambda_1\) is the constant in (6).

**Proof.** Let us multiply the equation (1) by \(u + u_t\) and integrate over \(\Omega\) then we get the relation

\[
\frac{d}{dt} \eta(t) + \|\Delta u\|^2 + \|u\|_{p+2}^{p+2} + \|u_t\|^2 + \|\Delta u_t\|^2
\]

\[= (\varphi + \varphi')f(t) + (u + u_t, \sum_{i=1}^n b_i u_{x_i x_i}) \quad (10)\]

where

\[
\eta(t) = \frac{1}{2}\|u(t)\|^2 + \|\Delta u(t)\|^2 + \frac{1}{p+2}\|u(t)\|_{p+2}^{p+2}.
\]

Replacing (8) into (10), we obtain

\[
\frac{d}{dt} \eta(t) + \|\Delta u\|^2 + \|u\|_{p+2}^{p+2} + \|u_t\|^2 + \|\Delta u_t\|^2
\]

\[= (\varphi + \varphi')(\varphi'(t) + (\Delta w, \Delta u) + (w, |u|^p u)
\]

\[-(w, \sum_{i=1}^n b_i u_{x_i x_i})] + (u + u_t, \sum_{i=1}^n b_i u_{x_i x_i}). \quad (11)\]

Using the inequalities (5) and (6) on the right-hand side of (11), we have the following estimates

\[|(u, \sum_{i=1}^n b_i u_{x_i x_i})| \leq \frac{B_0^2}{\lambda_1} \|\Delta u\|^2 \quad (12)\]

\[|(u_t, \sum_{i=1}^n b_i u_{x_i x_i})| \leq \|u_t\|^2 + \frac{B_0^2}{4} \|\Delta u\|^2 \quad (13)\]

\[|(\Delta w, \Delta u)(\varphi + \varphi')| \leq \frac{\xi}{2} \|\Delta u\|^2 + \frac{1}{\xi} \|\Delta w\|^2 (|\varphi|^2 + |\varphi'|^2) \quad (14)\]

\[|(w, \sum_{i=1}^n b_i u_{x_i x_i})(\varphi + \varphi')| \leq \frac{\xi}{2} \|\Delta u\|^2 + \frac{1}{\xi} \|\Delta w\|^2 (|\varphi|^2 + |\varphi'|^2) \quad (15)\]
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\[ + \frac{B_0}{\xi} \| \Delta w \|^2 (|\varphi|^2 + |\varphi'|^2) \]  

(15)

\[ |(w, u^p)(\varphi + \varphi')| \leq \xi |u(t)|^{|p+2 \atop p+2} \]

\[ + C(\xi, p) \| w \|_{p+2}^{|p+2 \atop p+2} (|\varphi|^2 + |\varphi'|^2). \]  

(16)

Rewriting (11) with estimates (12)-(16), we get the following differential inequality

\[ \frac{d}{dt} \eta(t) + \left( 1 - \xi - \frac{(4 + \lambda_1)B_0}{4\lambda_1} \right) \| \Delta u \|^2 \]

\[ + (1 - \xi) \| u \|_{p+2}^{|p+2 \atop p+2} \leq D(t) \]  

(17)

where

\[ D(t) = (|\varphi|^2 + |\varphi'|^2) \left( \xi \| \Delta w \|^2 + \frac{B_0}{\xi} \| \Delta w \|^2 + \langle \xi, p \rangle \| w \|_{p+2}^{|p+2 \atop p+2} \right). \]

We choose \( \xi_0 > 0 \) such that

\[ \xi_0 \leq \xi \leq 1 - \frac{(4 + \lambda_1)B_0}{4\lambda_1}, \]

and take

\[ K_1 = \min \left\{ \frac{\xi_0}{4} \left( 1 - \xi_0 - \frac{(4 + \lambda_1)B_0}{4\lambda_1} \right) \right\}. \]

So, (17) follows

\[ \frac{d}{dt} \eta(t) + K_1 (\frac{\xi_0}{4} \| \Delta u \|^2 + \| u \|_{p+2}^{|p+2 \atop p+2}) \leq D(t). \]  

(18)

The last term on the left-hand side of (18) can be written

\[ \frac{3}{2} \| \Delta u \|^2 + \| u \|_{p+2}^{|p+2 \atop p+2} \geq \lambda_1 \| u \|^2 + \| \Delta u \|^2 + \| u \|_{p+2}^{|p+2 \atop p+2} \]

\[ \geq K_2 \left( \frac{\lambda_1}{2} \| u \|^2 + \| \Delta u \|^2 + \frac{1}{p+2} \| u \|_{p+2}^{|p+2 \atop p+2} \right), \]  

(19)

where \( K_2 = \min \left\{ \lambda_1, \frac{1}{p+2} \right\} \). It follows from (18) and (19)

\[ \frac{d}{dt} \eta(t) + K_3 \eta(t) \leq D(t). \]  

(20)

where \( K_3 = K_1 K_2 \). After solving first-order differential inequality (20), it concludes that

\[ \lim_{t \to \infty} \left( \| \Delta u \|^2 + \frac{1}{p+2} \| u \|_{p+2}^{|p+2 \atop p+2} \right) = 0. \]

The proof is completed.

3. CONCLUSION

Long-time behaviour of the solutions to the inverse problem (1)-(4) is asymptotically stable. It means that \( \| \Delta u \| \) and \( \| u \|_{p+2} \) goes to zero as \( t \to \infty \).

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4. REFERENCES


