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An Approach to Neutrosophic Subrings

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Abstract

In this article we aim to construct some algebra on single valued neutrosophic sets. For this reason, we propose a new notion which is called a neutrosophic subring by combining the ring structure and neutrosophic sets. Then we establish some fundamental characteristics of the presented notion.

Keywords: neutrosophic set, single valued neutrosophic set, classical ring, homomorphism of rings.

1. INTRODUCTION

In human life situations, different types of uncertainties are encountered. Since the classical set is invalid to handle the described uncertainties, Zadeh [16] first gave the definition of a fuzzy set. According to this definition, a fuzzy set is a function described by a membership value which has taken degrees in a unit interval. But, later it has been seen that this definition is in adequate by consideration not only the membership degree but also the non-membership degree. So, Atanassov [2] described a new theory named as intuitionistic fuzzy set theory to handle mentioned ambiguity. Since this set have some problems in applications, Smarandache [14] introduced neutrosophy to solve the problems that involve indeterminate and inconsistent information. "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as the interactions with different ideational spectra"[14]. Neutrosophic set is a generalization both of the fuzzy set and intuitionistic fuzzy set, where all of the membership functions are represented independently in a different way of intuitionistic fuzzy set. Wang et al. [15] specified the definition of a neutrosophic set, named as a single valued neutrosophic set, to make more applicable the theory to real life problems. According to this definition, The single valued neutrosophic set (SVNS) is an extension of a classical set, (intuitionistic) fuzzy set, vague set and etc. Vasantha Kandasamy and Florentin Smarandache [8] discussed some algebraic structures on neutrosophic sets.

So far, the theory of SVNS is applied the direction on algebra and topology by some authors (see [1, 3, 4, 10, 12, 13]). Liu [9] defined the concept of a fuzzy ring. Later, Martinez [11] and Dixit et al.[5] studied on fuzzy ring and obtain certain ring theoretical analogue. Hur et al.[6] proposed the notion of an intuitionistic fuzzy subring. Vasantha Kandasamy and Florentin Smarandache [7] studied the neutrosophic rings.

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In this work, in a different direction from [7], we give an approach to a single valued neutrosophic subring of a classical ring as a continuation study of neutrosophic algebraic structures discussed in [4]. We define neutrosophic subring and also present some properties of this structure. Moreover, we examine homomorphic image and preimage of a neutrosophic subring. By this way, we obtain the generalized form of the fuzzy subring and intuitionistic fuzzy subring of a classical ring.

2. PRELIMINARIES

Throughout this section, \( X \) denotes the universal set which is non-empty.

**Definition 2.1** [14] A neutrosophic set \( N \) on \( X \) is defined by:

\[
N = \{(x, t_N(x), i_N(x), f_N(x)) : x \in X\}
\]

where \( t_N, i_N, f_N : X \to \] 0, 1]\ are functions satisfy the inequality \(-3 \leq t_N(x) + i_N(x) + f_N(x) \leq 3\)

According to the original definition, the neutrosophic set takes the value from real standard or non-standard subsets of \([-3, 3]\]. Since it is not appropriate to consider the degree which belongs to a real standard or a non-standard subset of \([-3, 3]\), in real life applications, especially in medical, engineering and statistical problems etc. For this reason, we prefer to deal with the following revised definition instead of the original definition of Smarandache.

**Definition 2.2** [15] A single valued neutrosophic set (SVNS) \( N \) on \( X \) is characterized by the truth-membership function \( t_N \); the indeterminacy-membership function \( i_N \) and the falsity-membership function \( f_N \). For each point \( x \) in \( X \); the values \( t_N(x), i_N(x), f_N(x) \) take place in the real unit interval \([0, 1]\).

In other words, \( N \) may be shown as

\[
N = \sum_{i=1}^{n}(t_N(x), i_N(x), f_N(x)) / x_i, x_i \in X.
\]

Since the membership functions \( t_N, i_N, f_N \) are defined from the universal set \( X \) into the unit interval \([0, 1]\) as \( t_N, i_N, f_N : X \to [0, 1] \), a (single-valued) neutrosophic set \( N \) will be denoted by a mapping described by \( N : X \to [0, 1] \)

\[
N(x) = (t_N(x), i_N(x), f_N(x)), \)

for simplicity. The family of all single-valued neutrosophic sets on \( X \) is denoted by \( SNS(X) \).

**Definition 2.3** [12, 15] Let \( N_1, N_2 \in SNS(X) \). Then

1. \( N_1 \) is contained in \( N_2 \); denoted as \( N_1 \subseteq N_2 \), if and only if \( N_1(x) \leq N_2(x) \). This means that \( t_{N_1}(x) \leq t_{N_2}(x) \), \( i_{N_1}(x) \leq i_{N_2}(x) \) and \( f_{N_1}(x) \geq f_{N_2}(x) \) Two sets \( N_1, N_2 \) are called equal, i.e., \( N_1 = N_2 \) iff \( N_1 \subseteq N_2 \) and \( N_2 \subseteq N_1 \).

2. the union of \( N_1 \) and \( N_2 \) is defined as \( N(x) = N_1(x) \lor N_2(x) \), where \( t_{N_1}(x) = t_{N_1}(x) \lor t_{N_2}(x) \), \( i_{N_1}(x) = i_{N_1}(x) \lor i_{N_2}(x) \), \( f_{N_1}(x) = f_{N_1}(x) \lor f_{N_2}(x) \), for each \( x \in X \).

3. the intersection of \( N_1 \) and \( N_2 \) is defined as \( N(x) = N_1(x) \land N_2(x) \), where \( t_{N_1}(x) = t_{N_1}(x) \land t_{N_2}(x) \), \( i_{N_1}(x) = i_{N_1}(x) \land i_{N_2}(x) \), \( f_{N_1}(x) = f_{N_1}(x) \lor f_{N_2}(x) \), for each \( x \in X \).

4. \( N^C \) denotes the complement of the SVNS \( N \) and it is defined by \( N^C(x) = (f_N(x), 1 - i_N(x), t_N(x)) \), for each \( x \in X \). Hence \( (N^C)^C = N \). The details of the set theoretical operations can be found in [12, 15].

**Definition 2.4** [4] Let \( g : X_1 \to X_2 \) be a mapping between classical sets, \( N_1 \in SNS(X_1) \) and \( N_2 \in SNS(X_2) \). Then the image \( g(N_1) \in SNS(X_2) \) and it is defined as follows.

\[
g(N_1)(x_1) = \left( g(t_{N_1}(x_2), i_{N_1}(x_2), f_{N_1}(x_2)) \right) = \left( g(t_{N_1}(x_2), g(i_{N_1}(x_2)), g(f_{N_1}(x_2)) \right),
\]

\[
\forall x_2 \in X_2, \text{where} \, g(t_{N_1})(x_2) = \{ \begin{cases} t_{N_1}(x_1), & \text{if } x_1 \in g^{-1}(x_2) \\ 0, & \text{otherwise} \end{cases} \}
\]
\[ g(i_{N_1})(x_2) = \begin{cases} \forall v_{N_1}(x_1), & \text{if } x_1 \in g^{-1}(x_2) \\ 0, & \text{otherwise} \end{cases} \]
\[ g(f_{N_1})(x_2) = \begin{cases} \forall f_{N_1}(x_1), & \text{if } x_1 \in g^{-1}(x_2) \\ 0, & \text{otherwise} \end{cases} \]

And the preimage \( g^{-1}(N_2) \in SNS(X_1) \) and it is defined as:
\[ g^{-1}(N_2)(x_1) = (t_{g^{-1}(N_2)}(x_1), i_{g^{-1}(N_2)}(x_1), f_{g^{-1}(N_2)}(x_1)) \]
\[ = (t_{N_2}(g(x_1)), i_{N_2}(g(x_1)), f_{N_2}(g(x_1))) \]
\[ = N_2(g(x_1)), \forall x_1 \in X_1. \]

## 3. NEUTROSOPHIC SUBRINGS

Now, we introduce the notion of neutrosophic subrings of a (classical) ring in a similar way of fuzzy case. We show that being a neutrosophic subring is preserved under a classical ring homomorphism. Also, we study some fundamental properties of a neutrosophic subring.

**Definition 3.1** Let \( H = (H, +, \cdot) \) be a classical ring and \( N \in SNS(H) \). Then \( N \) is called a neutrosophic subring of \( H \) if the following properties are satisfied for each \( x, h \in H \).

\( (H1) N(x + h) \geq N(x) \wedge N(h) \).
\( (H2) N(-h) \geq N(h) \).
\( (H3) N(x \cdot h) \geq N(x) \wedge N(h) \).

\( NSR(H) \) denotes the collection of all neutrosophic subrings of \( H \).

Throughout this study, \( H \) denotes a classical ring, unless otherwise specified.

**Example 3.2** Let us consider \( H = Z_4 = \{0, 1, 2, 3\} \) as the classical ring with the operations \( \oplus \) and \( \odot \) defined by \( r \oplus s = r + s \) and \( r \odot s = r \cdot s \) for all \( r, s \in Z_4 \), respectively. Define the neutrosophic set \( N \) on \( H \) as follows.

\[ N = \{0.8, 0.4, 0.1\}/0 + (0.5, 0.3, 0.5)/1 + (0.7, 0.4, 0.3)/2 + (0.5, 0.3, 0.5)/3 \]

It is easy to verify that the neutrosophic set defined above is a neutrosophic subring of \( H \).

**Theorem 3.3.** Let \( H \) be a fixed classical ring and \( N \in SNS(H) \). Then \( N \in NSR(H) \) iff the conditions given below hold.

1. \( (h_1 - h_2) \geq (h_1) \wedge (h_2), \) for all \( h_1, h_2 \in H \).
2. \( (h_1 \cdot h_2) \geq (h_1) \wedge (h_2), \) for all \( h_1, h_2 \in H \).

**Proof.** Let \( N \) be a neutrosophic subring of \( H \). Then the following inequality is valid from the conditions \((H1) \) and \((H2) \) as follows:

\[ N(h_1 - h_2) = N(h_1 + (-h_2)) \geq N(h_1) \wedge N(h_2). \]

Conversely, suppose that the conditions \((1) \) and \((2) \) are satisfied. Then the following is clearly obtained.

\[ N(0) = N(h_1 - h_1) \geq N(h_1) \wedge N(h_1) = N(h_1), \]

for each \( h_1 \in H \) (where \( 0 \) is the unit of the sum operation of \( H \)).

\[ N(-h_1) = N(0 - h_1) \geq N(0) \wedge N(h_1) \geq N(h_1) \wedge N(h_1) = N(h_1), \]

for each \( h_1 \in H \). By using these inequalities, we now obtain that

\[ N(h_1 + h_2) = N(h_1 - (-h_2)) \geq N(h_1) \wedge N(-h_2) \geq N(h_1) \wedge N(h_2). \]

**Theorem 3.4.** If \( N_1, N_2 \in SNS(H) \) are neutrosophic subrings of \( H \), then so the intersection \( N_1 \cap N_2 \) is.

**Proof.** Let \( h_1, h_2 \in H \) be arbitrary. By Theorem 3.3, we need to show that

\[ (N_1 \cap N_2)(h_1 - h_2) \geq (N_1 \cap N_2)(h_1) \wedge (N_1 \cap N_2)(h_2) \]

And

\[ (N_1 \cap N_2)(h_1 \cdot h_2) \geq (N_1 \cap N_2)(h_1) \wedge (N_1 \cap N_2)(h_2) \]

First consider the following

\[ t_{N_1 \cap N_2}(h_1 - h_2) = t_{N_1}(h_1 - h_2) \wedge t_{N_2}(h_1 - h_2) \]

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Proposition 3.6. Let \( N \in NSR(H) \) iff for each \( \beta \in [0,1] \), \( \beta \)-level sets of \( N \), \( (t_n)_{\beta} \), \( (i_n)_{\beta} \) and \( (f_n)_{\beta} \) are classical subrings of \( H \).

**Proof.** Let \( N \) be a neutrosophic subring of \( H, \beta \in [0,1] \) and \( h_1, h_2 \in (t_n)_{\beta} \) (similarly, \( h_1, h_2 \in (i_n)_{\beta} \)). By the assumption,

\[
t_n(h_1 - h_2) \geq t_n(h_1) \land t_n(h_2) \geq \beta \land \beta = \beta.
\]

The other inequalities are similarly proved for each \( h_1, h_2 \in H \).

Consequently, \( N_1 \cap N_2 \) is a neutrosophic subring of \( H \), as desired.

**Definition 3.5.** [4] Let \( N \in SNS(X) \) and \( \beta \in [0,1] \) be given. Then the level sets, which are classical sets on \( X \), of \( N \) are defined in a following way.

\[
(t_n)_{\beta} = \{ x \in X \mid t_n(x) \geq \beta \},
\]

\[
(i_n)_{\beta} = \{ x \in X \mid i_n(x) \geq \beta \}
\]

and

\[
(f_n)_{\beta} = \{ x \in X \mid f_n(x) \leq \beta \}.
\]

Following results are easily proved by using Definition 3.5.

(1) If \( N \subseteq M \) and \( \beta \in [0,1] \), then \( (t_n)_{\beta} \subseteq (t_m)_{\beta} \), \( (i_n)_{\beta} \subseteq (i_m)_{\beta} \) and \( (f_n)_{\beta} \subseteq (f_m)_{\beta} \).

(2) \( \beta \leq \gamma \) implies \( (t_n)_{\beta} \subseteq (t_n)_{\gamma} \), \( (i_n)_{\beta} \subseteq (i_n)_{\gamma} \) and \( (f_n)_{\beta} \subseteq (f_n)_{\gamma} \).

Theorem 3.7. Let \( H_1 \) and \( H_2 \) be two classical rings and \( g: H_1 \to H_2 \) be a ring homomorphism. If \( N \) is a neutrosophic subring of \( H_1 \), then \( g(N) \), the image of \( N \), is a neutrosophic subring of \( H_2 \).

**Proof.** Suppose that \( N \) is a neutrosophic subring of \( H_1 \) and \( k_1, k_2 \in H_2 \). If \( g^{-1}(k_1) = \varnothing \) or \( g^{-1}(k_2) = \varnothing \), then it is obvious that \( g(N) \) is a neutrosophic subring of \( H_2 \). Assume that there exist \( h_1, h_2 \in H_1 \) such that \( g(h_1) = k_1 \) and \( g(h_2) = k_2 \). Since \( g \) is a homomorphism of rings,

\[
g(h_1 - h_2) = g(h_1) - g(h_2) = k_1 - k_2
\]

and

\[
g(h_1, h_2) = g(h_1), g(h_2) = k_1, k_2.
\]
In similar computations, it is seen that
\[ g(N)(k_1 - k_2) \geq g(N)(k_1) \wedge g(N)(k_2). \]
By using the above inequalities, we now prove that
\[ g(N)(k_1 - k_2) \geq g(N)(k_1) \wedge g(N)(k_2). \]

So, the followings become valid:
\[
\begin{align*}
g(t_N)(k_1 - k_2) &= \bigvee_{k_1 - k_2 = g(h)} t_N(h), \\
g(i_N)(k_1 - k_2) &= \bigvee_{k_1 - k_2 = g(h)} i_N(h), \\
g(f_N)(k_1 - k_2) &= \bigwedge_{k_1 - k_2 = g(h)} f_N(h),
\end{align*}
\]

Hence being a neutrosophic subring is preserved under a homomorphism of rings.

**Theorem 3.8.** Let \( H_1 \) and \( H_2 \) be two classical rings and \( g:H_1 \to H_2 \) be a homomorphism of rings. If \( M \in NSR(H_2) \), then the preimage \( g^{-1}(M) \in NSR(H_1) \).

**Proof.** Suppose that \( M \) is a neutrosophic subring of \( H_2 \) and \( h_1, h_2 \in H_1 \). Since \( g \) is a homomorphism of rings, then the following inequality is obtained:
\[
g^{-1}(M)(h_1 - h_2) = (t_M(g(h_1 - h_2)), i_M(g(h_1 - h_2)), f_M(g(h_1 - h_2)))
\]
\[
= (t_M(g(h_1) - g(h_2)), i_M(g(h_1) - g(h_2)), f_M(g(h_1) - g(h_2))) \\
\geq (t_M(g(h_1)), i_M(g(h_1)), f_M(g(h_1)) \wedge f_M(g(h_2))) \\
= g^{-1}(M)(h_1) \wedge g^{-1}(M)(h_2)
\]
In similar computations, it is clear that \( g^{-1}(M)(h_1, h_2) \geq g^{-1}(M)(h_1) \wedge g^{-1}(M)(h_2) \).

Therefore, \( g^{-1}(M) \) is a neutrosophic subring of the classical ring \( H_1 \).

### 4. CONCLUSION

The concept of a ring is one of the fundamental structures in algebra. The problem of classifying all rings (in a given class) up to homomorphism is far more complicated than the corresponding problem for groups. A single-valued neutrosophic set is a kind of neutrosophic set which is suitable to use in real world applications. So, we decided to combine these concepts and to propose the definition of a neutrosophic subring of a given crisp ring, in the direction of [4], and observe its fundamental properties. For further research, one can handle cyclic (respectively, symmetric, abelian) neutrosophic group structure.
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