Title: On some Zweier convergent vector valued multiplier spaces

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Received: 2018-12-06 00:00:00
Accepted: 2018-12-31 00:00:00

Article Type: Research Article
Volume: 23
Issue: 4
Month: August
Year: 2019
Pages: 541-548

How to cite
Ramazan Kama; (2019), On some Zweier convergent vector valued multiplier spaces.
Sakarya University Journal of Science, 23(4), 541-548, DOI: 10.16984/saufenbilder.492788
Access link
http://www.saujs.sakarya.edu.tr/issue/43328/492788

New submission to SAUJS
http://dergipark.gov.tr/journal/1115/submission/start
On Zweier convergent vector valued multiplier spaces

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Abstract

In this paper, we introduce the Zweier convergent vector valued multiplier spaces $M_Z^\infty (\sum T_i x_i)$ and $M_{w2}^\infty (\sum T_i x_i)$. We study some topological and algebraic properties on these spaces. Furthermore, we study some inclusion relations concerning these spaces.

Keywords: vector valued multiplier space, Zweier matrix, summing operator, operator valued series.

1. INTRODUCTION

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. We shall denote the space of all real valued sequences by $w = \{ x = (x_i) : x_i \in \mathbb{R} \}$.

Any vector subspace of $w$ is called as a sequence space. Let $l_\infty, c$ and $c_0$ denote the spaces of all bounded, convergent and null sequences $x = (x_i)$ with real terms, respectively, normed by $\| x \|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

A sequence space $X$ with linear topology is called a $K$-space provided each of the maps $p_i : X \to \mathbb{R}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. If $x \in X$, then $e^i \otimes x$ denote the sequence with $x$ in the $i^{th}$ coordinate and zero in the other coordinates. If $\mathfrak{I} \subset \mathbb{N}$, $\chi_\mathfrak{I}$ denote the characteristic function of $\mathfrak{I}$ and $x = (x_i)$ is any sequence, $\chi_\mathfrak{I}x$ denote the coordinatewise product of $\chi_\mathfrak{I}$ and $x$. A sequence space $X$ is monoton if $\chi_\mathfrak{I}x \in X$ for every $\mathfrak{I} \subset \mathbb{N}$ and $x \in X$.

Let $X$ and $Y$ be sequence spaces and $A = (a_{ni})$ be an infinite matrix of real numbers $a_{ni}$, where $n,i \in \mathbb{N}$. Then, we say that $A$ defines a matrix mapping from $X$ to $Y$. If for every sequence $x = (x_i) \in X$ the sequence $Ax = (Ax_n)$, the $A-$ transform of $x \in X$ in $Y$, where $(Ax)_n = \sum k a_{ni} x_i$ for each $n \in \mathbb{N}$. The matrix domain $X_A$ of an infinite matrix $A$ in a sequence space $X$ is defined by

$X_A = \{ x = (x_i) \in w : Ax \in X \}$

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which is a sequence space [4, 6, 11].

Şengönül [15] defined the sequence \( y = (y_k) \) which is frequently used as the \( Z^\alpha \)-transformation of the sequence \( x = (x_k) \) i.e.

\[
y_k = ax_k + (1 - a)x_{k-1},
\]

where \( x_{-1} = 0, 1 < k < \infty \) and \( Z^\alpha \) denotes the matrix \( Z^\alpha = (z_{ij}) \) defined by

\[
(z_{ij}) = \begin{cases} 
\alpha, & \text{if } i = j, \\
1 - \alpha, & \text{if } i - 1 = j, \\
0, & \text{otherwise}. 
\end{cases}
\]

Following Başar and Altay [5], Şengönül [15] introduced the Zweier sequence spaces \( Z \) and \( Z_0 \) as follows:

\[
Z = \{ x = (x_k) \in w : Z_p x \in c \},
\]

\[
Z_0 = \{ x = (x_k) \in w : Z_p x \in c_0 \}.
\]

For details on Zweier sequence spaces we also refer to [8–10].

Let \( X, Y \) be normed spaces, \( L(X, Y) \) be also the space of continuous linear operators from \( X \) into \( Y \) and \( \sum T_i \) be a series in \( L(X, Y) \), \( \lambda \) be a vector space of \( X \)-valued sequences which contains \( c_0(X) \), the space of all sequences which are eventually 0. By \( l_\infty(X) \) and \( c_0(X) \), we denote the \( X \)-valued sequence spaces of bounded and convergence to zero, respectively. The series \( \sum T_i \) is \( \lambda \)-multiplier convergent if the series \( \sum T_i x_i \) converges in \( Y \) for every sequence \( x = (x_i) \in \lambda \). The series \( \sum T_i \) is \( \lambda \)-multiplier Cauchy if the series \( \sum T_i x_i \) is Cauchy in \( Y \) for every sequence \( x = (x_i) \in \lambda \). For more information about vector valued multiplier spaces and multiplier convergent series, see [2, 7, 8, 13].

Let \( \sum T_i \) be a series in \( L(X, Y) \). Then, we will define the spaces

\[
M^\infty_w(\sum T_i x_i) = \{ x = (x_i) \in l_\infty(X) : Z - \sum T_i x_i \text{ exists} \}
\]

and

\[
M^\infty_w(\sum T_i x_i) = \{ x = (x_i) \in l_\infty(X) : Z - \sum T_i x_i \text{ exists} \}
\]

endowed sup norm, where

\[
Z - \sum T_i x_i = \lim_{n \to \infty} (1 - \alpha) \sum_{i=1}^{n-1} T_i x_i^\alpha + \alpha \sum_{i=1}^{n} T_i x_i^\alpha
\]

and

\[
wZ - \sum T_i x_i = \lim_{n \to \infty} (1 - \alpha) \sum_{i=1}^{n-1} f(T_i x_i^\alpha) + \alpha \sum_{i=1}^{n} f(T_i x_i^\alpha)
\]

\( f \in Y^* \) (dual of \( Y \)). Notice that \( M^\infty_w(\sum T_i x_i) \subset M^\infty_w(\sum T_i x_i) \subset l_\infty(X) \).

In [1, 12], authors introduced some subspaces of \( l_\infty \) by means of multiplier convergent series and studied some properties of this spaces. Also, in [3, 14], the above spaces studied in the case of some convergence.

In this paper, we will show that the spaces \( M^\infty(\sum T_i x_i) \) and \( M^\infty_w(\sum T_i x_i) \) are Banach spaces by means of \( c_0(X) \)-multiplier convergent series. Also, we will give some characterizations of \( l_\infty(X) \) and \( c_0(X) \)-multiplier convergent series by using summing operators related to the series \( \sum T_i \).

2. THE ZWEIER SUMMABILITY SPACE

Before starting this section, we give the following proposition will be used for establishing some results of this study:

**Proposition 2.1.** \( \sum T_i \) \( c_0(X) \)-multiplier convergent series if and only if the set
\[ E = \left\{ \sum_{i=1}^{n} T_i x_i : \|x_i\| \leq 1, n \in \mathbb{N} \right\} \]  

is bounded [14].

The following theorem gives the completeness of the space \( M_∞^p(\sum_i T_i x_i) \).

**Theorem 2.2.** Let \( X \) and \( Y \) are normed spaces and \( \sum_i T_i \) is a series in \( L(X,Y) \). If

(i) \( X \) and \( Y \) are Banach spaces,

(ii) The series \( \sum_i T_i \) c\(_0\)(\(X\)) - multiplier convergent,

then \( M_∞^p(\sum_i T_i x_i) \) is a Banach space.

**Proof.** Since the series \( \sum_i T_i \) c\(_0\)(\(X\)) - multiplier convergent, by Proposition 2.1, there exists \( M > 0 \) such that

\[ M = \sup \left\{ \left\| \sum_{i=1}^{n} T_i x_i \right\| : \|x_i\| \leq 1, n \in \mathbb{N} \right\}. \]

We suppose that \((x^m)\) be a Cauchy sequence in \( M_∞^p(\sum_i T_i) \). Since \( M_∞^p(\sum_i T_i) \subset l_∞(X) \) and \( l_∞(X) \) is a Banach space (since \( X \) is a Banach space), there exists \( x = (x_i^0) \in l_∞(X) \) such that \( \lim_{m \to \infty} x^m = x^0 \). We will show that \( x^0 \in M_∞^p(\sum_i T_i) \).

We take \( \epsilon > 0 \). Then, there exists \( m_0 \in \mathbb{N} \) such that

\[ \|x^m - x^0\| < \frac{\epsilon}{3M} \]

for \( m \geq m_0 \). Since \( \frac{3M}{\epsilon} \|x^m - x^0\| < 1, \)

\[ \frac{3M}{\epsilon} \left\| (1 - \alpha) \sum_{i=1}^{n-1} T_i x_i^m - x_i^0 \right\| + \alpha \sum_{i=1}^{n} T_i x_i^m - x_i^0 \right\| \leq M \]

and so

\[ \left\| (1 - \alpha) \sum_{i=1}^{n-1} T_i x_i^m - x_i^0 \right\| + \alpha \sum_{i=1}^{n} T_i x_i^m - x_i^0 \right\| \leq \frac{\epsilon}{3} \]

for \( m \geq m_0 \) and \( n \in \mathbb{N} \). On the other hand, since \((x^m)\) is a Cauchy sequence in \( M_∞^p(\sum_i T_i) \) there exists sequence \((y_m) \subset Y \) such that

\[ \left\| (1 - \alpha) \sum_{i=1}^{n-1} T_i x_i^m + \alpha \sum_{i=1}^{n} T_i x_i^m - y_m \right\| < \frac{\epsilon}{3} \]

for \( n \geq n_0 \). If we take \( p > q \geq m_0 \), from (2) and (3), then we have \( \|y_p - y_q\| < \epsilon \). Hence, \((y_m)\) is a Cauchy sequence. Let \( y_m = y_0 \) and suppose that \( \|y_m - y_0\| < \frac{\epsilon}{3} \). Consequently,

\[ \left\| (1 - \alpha) \sum_{i=1}^{n-1} T_i x_i^m + \alpha \sum_{i=1}^{n} T_i x_i^m - y_0 \right\| \leq \left\| (1 - \alpha) \sum_{i=1}^{n-1} T_i x_i^m - x_i^0 \right\| + \alpha \sum_{i=1}^{n} T_i x_i^m - x_i^0 \right\| + \left\| (1 - \alpha) \sum_{i=1}^{n-1} T_i x_i^m + \alpha \sum_{i=1}^{n} T_i x_i^m - y_m \right\| + \|y_m - y_0\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \]

for \( n \geq n_0 \). This means that \( x^0 \in M_∞^p(\sum_i T_i) \).

In the next theorem we show that the converse of above theorem is hold. But, it does not need to be the spaces \( X \) and \( Y \) are complete.

**Theorem 2.3.** If \( M_∞^p(\sum_i T_i) \) is a Banach space, then \( \sum_i T_i \) c\(_0\)(\(X\)) - multiplier convergent series.

**Proof.** We consider the sequence \( x = (x_i) \in c_0(X) \). From the closedness of \( M_∞^p(\sum_i T_i) \) and

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Then the series $\sum_i T_i x_i$ is subseries Zweier convergent because of $c_0(X)$ is a monoton space. So, $\sum_i T_i x_i$ is weakly subseries Zweier convergent series. Using Orlicz-Pettis theorem ([1, Theorem 4.1]), we obtain that the series $\sum_i T_i x_i$ is subseries norm convergent, and hence $\sum_i T_i$ is $c_0(X)$ multiplier convergent.

**Remark 2.4.** (1) In Theorem 2.2, if $Y$ is not a Banach space, then there exists a sequence $y = (y_i)$ in $Y$ and $F \in Y^* \setminus Y$ such that

$$||y|| < \frac{1}{3^i} \text{ and } \sum_i y_i = F$$

for every $i \in \mathbb{N}$. Also, note that $Z - \sum_i y_i = F$. We take $x_0 \in X$ with $||x_0|| = 1$. By Hahn-Banach theorem, we choose $x_0^* \in X^*$ such that $x_0^*(x_0) = ||x_0||$. We denote sequence $T_i \in L(X,Y)$ by $T_i x = x_0^* (x) 3^i y_i$ for each $i \in \mathbb{N}$. It is obtain that $\sum_i T_i$ is $c_0(X)$ multiplier Cauchy. Consider the sequence $x = (x_0/3^i)$ in $c_0(X)$. Then $x^n = \sum_{i=1}^n e^i \otimes x_0 /3^i \in M_Z^\infty (\sum_i T_i)$ for every $n \in \mathbb{N}$ and $x^n \to x_0/3^i$, but since

$$Z - \sum_i T_i x_i = Z - \sum_i \frac{1}{3^i} x_0^* (x_0) 3^i y_i = Z - \sum_i y_i = F,$$

$M_Z^\infty (\sum_i T_i)$ is not a Banach space.

(2) It is well know that if $\lim x_i = x_0$, then $Z - \lim x_i = x_0$, and also $\sum_i x_i = x_0$, then $Z - \sum_i x_i = x_0$. Therefore, if

$$M^\infty (\sum_i T_i) = \left\{ x = (x_i) \in l_\infty(X) : \exists T_i x_i \text{ exists} \right\},$$

then we obtain the inclusion $M^\infty (\sum_i T_i) \subset M_Z^\infty (\sum_i T_i)$.

(3) Let $X$ and $Y$ be normed spaces. We denote the summing operator associate with the series $\sum_i T_i$

$$S: M_Z^\infty (\sum_i T_i) \to Y, \quad S(x) = Z - \sum_i T_i x_i.$$

Then, the summing operator $S$ is continuous if and only if the series $\sum_i T_i$ is $c_0(X)$ multiplier Cauchy. Let us suppose that $S$ is continuous. Since $c_0(X) \subset M_Z^\infty (\sum_i T_i)$, and if $x = (x_i) \in c_0(X)$ with $||x|| \leq 1$ such that $x_i = 0$ for all $i > k$, we have that

$$||S_1 x_1 + \cdots + S_k x_k|| = ||Sx|| \leq ||S||.$$

Therefore

$$\sup_k \left\{ \left\| \sum_{i=1}^k T_i x_i \right\| : \left\| x_i \right\| \leq 1, k \in \mathbb{N} \right\} \leq ||S||,$$

and hence, the series $\sum_i T_i$ is $c_0(X)$ multiplier Cauchy by Proposition 2.1.

Now, suppose that $\sum_i T_i$ is $c_0(X)$ multiplier Cauchy. Then, by Proposition 2.1, the set $E = \{ ||\sum_{i=1}^k T_i x_i|| : ||x_i|| \leq 1, k \in \mathbb{N} \}$ is bounded. We take $||e|| \leq K$ for every $e \in E$. Let $x = (x_i) \in M_Z^\infty (\sum_i T_i)$ with $||x|| \leq 1$. Thus $Z - \sum_{i=1}^k T_i x_i$ exists, and hence

$$||S_k(x)|| = \left\| Z - \sum_{i=1}^k T_i x_i \right\| \leq K$$

for $k \in \mathbb{N}$. This means that $S$ is continuous.

(4) We suppose that $Y$ is a Banach space. Then, we will show that the summing operator $S$ is compact if and only if the series $\sum_i T_i$ is $l_\infty(X)$ multiplier convergent. Indeed, let $S$ be compact and $x = (x_i) \in l_\infty(X)$. If we define the following set that is bounded on the space $M_Z^\infty (\sum_i T_i)$

$$M = \left\{ \sum_{i \in \mathbb{N}} e^i \otimes x_i : \exists \text{ is finite, } \left\| x_i \right\| \leq 1 \right\},$$

then $S(M) = Z - \sum_{i \in \mathbb{N}} T_i x_i$ is finite, $||x_i|| \leq 1$ is relatively compact. Hence, the series $\sum_i T_i x_i$ is subseries norm Zweier summability ([13, Theorem 2.48]), and so the series $\sum_i T_i x_i$ is
subseries norm convergent by Orlicz-Pettis theorem. That is \( \sum_i T_i \) is \( l_\infty(X) \) – multiplier convergent series.

Conversely, let \( \sum_i T_i \) is \( l_\infty(X) \) – multiplier convergent series, then \( Z - \sum_i T_i x_i \) is uniformly convergent series for \( \|x_i\| \leq 1 \) ([13, Corollary 11.11]). If we define the operators \( S_n: M_\infty^w(\sum_i T_i) \rightarrow Y \) by \( S_n(x) = Z - \sum_{i=1}^n T_i x_i \) for \( n \in \mathbb{N} \), then

\[
\|S_n - S\| = \left\| Z - \sum_{i=1}^n T_i x_i - Z - \sum_{i=n+1}^\infty T_i x_i \right\|
= \left\| Z - \sum_{i=n+1}^\infty T_i x_i \right\| \rightarrow 0
\]

for \( \|x_i\| \leq 1 \), as \( n \rightarrow \infty \). Therefore, \( S \) is compact.

By Theorem 2.2, Theorem 2.3 and Remark 2.4, we can obtain the following corollary:

**Corollary 2.5.** If \( X \) and \( Y \) are Banach spaces and \( \sum_i T_i \) is a series in \( L(X,Y) \), then the following statements are equivalent:

(i) \( \sum_i T_i \) \( c_0(X) \) – multiplier convergent series.

(ii) \( M_\infty \sum_i T_i \) is a Banach space.

(iii) \( c_0(X) \subseteq M_\infty \sum_i T_i \).

(iv) \( M_\infty^w(\sum_i T_i) \) is a Banach space.

(v) \( c_0(X) \subseteq M_\infty^w(\sum_i T_i) \).

3. THE WEAK ZWEIER SUMMABILITY SPACE

In this section, we will extend that to the space \( M_\infty^w(\sum_i T_i) \) some of the conclusions obtained in the preceding section for the space \( M_\infty^w(\sum_i T_i) \). We begin this section by the following theorem.

**Theorem 3.1.** If \( X \) and \( Y \) are Banach spaces and the series \( \sum_i T_i \) \( c_0(X) \) – multiplier convergent, then \( M_\infty^w(\sum_i T_i x_i) \) is a Banach space.

**Proof.** Let \( (x^m) \subset M_\infty^w(\sum_i T_i x_i) \) be a Cauchy sequence. Then, \( \lim_{m \to \infty} x^m = x^0 \) in \( l_\infty(X) \). We will prove that \( x^0 \in M_\infty^w(\sum_i T_i) \).

If the proof of Theorem 2.2 is followed, then there exists \( m_0 \in \mathbb{N} \) such that

\[
\left\| (1 - \alpha) \sum_{i=1}^{n-1} T_i (x_i^m - x_i^0) + \alpha \sum_{i=1}^n T_i (x_i^m - x_i^0) \right\| < \frac{\varepsilon}{3}
\]

for \( m \geq m_0 \) and \( n \in \mathbb{N} \). If \( p > q \geq m_0 \) are fixed, then a functional \( f \in S_Y^* \) (unit sphere in \( Y^* \)) can be found such that \( \|y_p - y_q\| = |f(y_p) - f(y_q)| \).

Since \( (x^m) \) is a Cauchy sequence in \( M_\infty^w(\sum_i T_i) \), there exists sequence \( (y_m) \subset Y \) such that

\[
\left\| (1 - \alpha) \sum_{i=1}^{n-1} f(T_i x_i^m) + \alpha \sum_{i=1}^n f(T_i x_i^m) - f(y_m) \right\| < \frac{\varepsilon}{3}
\]

for \( n \geq n_0 \). From (4) and (5), we have \( \|y_p - y_q\| < \varepsilon \). Thus, \( (y_m) \) is a Cauchy sequence. Since \( Y \) is a Banach space, there exists \( y_0 \in Y \) such that \( \|y_m - y_0\| < \frac{\varepsilon}{3} \). Finally, we obtain that the following inequalities,

\[
\left| (1 - \alpha) \sum_{i=1}^{n-1} f(T_i x_i^0) + \alpha \sum_{i=1}^n f(T_i x_i^0) - f(y_0) \right| \\
\leq \left| (1 - \alpha) \sum_{i=1}^{n-1} f(T_i (x_i^m - x_i^0)) + \alpha \sum_{i=1}^n f(T_i (x_i^m - x_i^0)) \right|
\]

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As we did Remark 2.4 (3), one can see that the summing operator $S$ is continuous if and only if the series $\sum_i T_i$ is $c_0(X) -$ multiplier Cauchy.

(4) Let $Y$ be a Banach space. If $S$ is compact, from Remark 2.4 (4), then the set $S(M)$ is weakly relatively compact, and hence $\sum_i T_i$ is $l_\infty(X) -$ multiplier convergent series. On the other hand, let us suppose that $Y$ is complete and the series $\sum_i T_i$ is $l_\infty(X) -$ multiplier convergent. Then, $wZ - \sum_i T_i x_i$ is uniformly convergent for $x_n = 1$ ([13, Corollary 11.11]). Therefore, we have that

$$\|S_n - S\| = \left\|wZ - \sum_{i=1}^n T_i x_i - wZ - \sum_{i=n+1}^\infty T_i x_i\right\| \rightarrow 0$$

for $\|x_n\| \leq 1$, as $n \rightarrow \infty$, where the operators $S_n: M^\infty_{wZ}(\sum_i T_i) \rightarrow Y$ is defined by $S_n(x) = wZ - \sum_{i=1}^n T_i x_i$ for $n \in N$. This implies that $S$ is compact.

By the previous theorems and remark above, we can give the following corollaries:

**Corollary 3.4.** If $X$ and $Y$ are Banach spaces and $\sum_i T_i$ is a series in $L(X,Y)$, then the following conditions are equivalent:

(i) $\sum_i T_i$ is $c_0(X) -$ multiplier convergent series.

(ii) $M^\infty_w \sum_i T_i$ is a Banach space.

(iii) $c_0(X) \subseteq M^\infty_w \sum_i T_i$.

(iv) $M^\infty_{wZ}(\sum_i T_i)$ is a Banach space.

(v) $c_0(X) \subseteq M^\infty_{wZ}(\sum_i T_i)$.

**Corollary 3.5.** If $Y$ is Banach space, then the following are equivalent:

(i) $S$ is compact.

(ii) $S$ is a weakly compact.

(iii) $\sum_i T_i$ is $l_\infty(X) -$ multiplier convergent series.
Finally, we will give a sufficient condition for the equivalence of both spaces, which are defined in the introduction.

**Proposition 3.6.** Let $X$ and $Y$ be normed spaces. If $\sum T_i x_i$ is $l_\infty(X) -$ multiplier Cauchy series, $M^\infty_Z(\sum T_i) = M^\infty_{wZ}(\sum T_i)$.

**Proof.** We prove that the inclusion $M^\infty_{wZ}(\sum T_i) \subset M^\infty_Z(\sum T_i)$ hold. If we take $x = (x_i) \in M^\infty_{wZ}(\sum T_i)$, then there exists $y_0 \in Y$ such that
\[
Z - \sum_i f(T_i x_i) = f(y_0)
\]
for every $f \in Y^\ast$. Also, since the series $\sum T_i x_i$ is $l_\infty(X) -$ multiplier Cauchy, the series $\sum T_i x_i$ is Cauchy in $Y$. Thus, there exists $F \in Y^{**}$ such that
\[
Z - \sum_i T_i x_i = F.
\]
If consider the uniqueness of limit, then we have $F = y_0$. Thus, $x = (x_i) \in M^\infty_Z(\sum T_i)$.

**Acknowledgments**

This work was supported by the Siirt University Research Fund with Project Number 2018-SİÜEĞİT-041.

4. REFERENCES


