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A Characterization of Some Class Nonlinear Eigenvalue Problem in VELS

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Abstract

In last the quarter century, many researchers have been interested by the theory of the variable exponent function space and its applications. We well-know that a normal mode analysis of a vibrating mechanical or electrical system gives rise to an eigenvalue problem. We will investigate a characterization of some class nonlinear eigenvalue problem in variable exponent Lebesgue spaces.

Keywords: Variable exponent, operator theory, Lebesgue spaces

1. INTRODUCTION

In this paper, we derive a new boundedness and compactness result for the Hardy operator in variable exponent Lebesgue spaces (VELS), $L^{p(.)}(0, l)$. A maximally weak condition is assumed on the exponent function. The last time, such a study was carry out in [1,2,3,4,5,6,7,8,9,13,14]. For a study the Dirichlet problem of some class nonlinear eigenvalue problem with nonstandard growth condition the obtained results is applied. Such equations arise in the studies of the so called Winslow effect physical phenomena [11] in the smart materials. In this connection, we mention recent studies for the multidimensional cases with application of Ambrosetti-Rabinoviche’s Mountain pass theorem approaches (see, e.g. in [1,10, 12]).

Theorem 1.1. Let $q,p : (0, l) \rightarrow (1, \infty)$ be measurable functions with $q(x) \geq p(x)$ on $(0, l)$. Assume $p$ be monotony increasing and the function $x^{-\frac{1}{p(x)+\delta}}$ is almost decreasing on $(0, l)$.

Then operator H boundedly acts the space $L^p(0, l)$ into $L^{q(.)}$. $x^{-\frac{1}{p(.)}} q(.)(0, l)$. Moreover, the norm of mapping depends on $p^-, p^+, \delta, \beta$.

In the given assertions, $L^{p,\alpha}(0, l)$ denotes the space of measurable functions with finite norm $\|xy^{\alpha}\|_{L^{p(.)}(0, l)}$, while $W^{1,\alpha}_{p(.)}(0, l)$ stands the space of absolutely continuous functions $y$ with $y(0) = 0$ and finite norm $\|y\|_{W^{1,\alpha}_{p(.)}} = \|y'\|_{L^{p(.)}}$.

We say, the function $\alpha : (0, l) \rightarrow (0, \infty)$ is almost increasing (decreasing) if there exists a constant $C > 0$ such that for any $0 < t_1 < t_2 < l$ it holds $\alpha(t_1) \leq C \alpha(t_2)$ ( $\alpha(t_1) \geq C \alpha(t_2)$ )

We need the following assertion.

Lemma 1.2. Let $p(x)$ be increasing for $x \in (0, l)$. Let $t \in A_n(x) = (2^{-n-1}x, 2^{-n}x]$. Then it holds $t^{-\frac{1}{p(x)}} \leq C t^{-\frac{1}{p_{x,n}^{(l)}}}$, where $p_{x,n} = \inf_{t \in A_n(x)} p(t)$.

Proof. Let $y \in A_n(x)$ be a point with $t^{-\frac{1}{p(y)}} \leq 2t^{-\frac{1}{p_{x,n}^{(l)}}}$. Let $y < t$ and both lie in $A_n(x)$. Then using almost decreasing of $x^{-\frac{1}{p+y}}$ it follows that

$t^{-\frac{1}{p(y)}} \leq cy^{-\frac{1}{p+y}}$

Using $t, y \in A_n(x)$, $(p_{x,n}^{(l)})' > 1$ it follows

$t^{-\frac{1}{p(y)}} \leq 2^{+\epsilon} C y^{-\frac{1}{p(y)}} \leq 2^{+\epsilon} C t^{-\frac{1}{p_{x,n}^{(l)}}}$
Now let $y > t$, then using increasing of $p$, $\frac{1}{p'}$ also will be increasing. Since $\frac{1}{p'(t)} < \frac{1}{p'(y)}$, it follows that
\[
\left( \frac{1}{p'(t)} \right)^{\frac{1}{p'(t)}} \leq C \left( \frac{1}{p'(y)} \right)^{\frac{1}{p'(y)}} \leq 2C t^{-\frac{1}{p'(x_n,y)}},
\]
where $c = \left( \frac{1}{p'} \right)^{\frac{1}{p'}} + \frac{1}{p'}$.

The Lemma 1.2 has been proved.

Proof of Theorem 1.1. Let $f : (0, l) \to (0, \infty)$ be a positive measurable function. It holds the identity

\[
Hf(x) = \sum_{n=1}^{\infty} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt
\]

Assume $\|f\|_p = 1$. Using the triangle property of $p(.)$ - norms

\[
\|x^\alpha Hf\|_q \leq \sum_{n=1}^{\infty} \left\| x^\alpha \int_{A_n(x)} f(t) dt \right\|_q
\]

With
\[
\alpha(x) = -\frac{1}{p'(x)} - \frac{1}{q(x)} \quad (\text{recall } A_n(x) = (2^{-n-1}x, 2^{-n}x))
\]

Derive estimation for every summand in (3). In this purpose get estimation for the proper modular

\[
l_q \left( x^{\alpha(\cdot)} \int_{A_n(x)} f(t) dt \right) = \int_0^l \left( x^{\alpha(x)} \int_{A_n(x)} f(t) dt \right)^q dx.
\]

Applying the assumption on $p$ (decreasing of $\frac{1}{p'} + \varepsilon$), and using the expression for

\[
q(x) = \frac{1}{-\alpha - \frac{1}{p'(x)}} \quad \text{we have}
\]

\[
l_q \left( x^{\alpha(x)} \int_{A_n(x)} f(t) dt \right) = \frac{\int_0^l \left( x^{\frac{1}{p'} + \varepsilon} \int_{A_n(x)} f(t) dt \right)^q dx}{x^{1+\varepsilon q(x)}}
\]

\[
\leq C q^* 2^{-n\varepsilon q} \int_0^l \frac{dx}{x} \left( \int_{A_n(x)} f(t) t^{-\frac{1}{p'(t)}} dt \right)^q(x)
\]

Therefore, from (3) using Holder’s inequality, it follows

\[
l_q \left( x^{\alpha(\cdot)} \int_{A_n(x)} f(t) dt \right) \leq C q^* 2^{-n\varepsilon q} \int_0^l \frac{dx}{x} \left( \int_{A_n(x)} f(t)^{p(x)} dt \right)^q \left( \frac{p(x)}{p(x)} \right)
\]

Applying this Lemma 1 and estimate (1) it follows from (6) that

\[
l_q \left( x^{\alpha(\cdot)} \int_{A_n(x)} f(t) dt \right) \leq \int_0^l \frac{dx}{x} \left( \int_{A_n(x)} f(t)^{p(x)} dt \right)^q \left( \frac{p(x)}{p(x)} \right) (c \ln 2) q^* 2^{-n\varepsilon q} C q^*
\]

Since

\[
\int_{A_n(x)} f(t)^{p(x)} dt \leq \int_{A_n(x)} f(t)^{p(t)} dt + \int_{A_n(x)} dt \leq 1 + 2^{-n} x \leq 1 + 2^{-n} l \leq l + 1.
\]

it follows

\[
l_q \left( x^{\alpha(\cdot)} \int_{A_n(x)} f(t) dt \right) \leq (c \ln 2) q^* 2^{-n\varepsilon q}.
\]

\[
\cdot \int_0^l \frac{dx}{x} \left( \frac{1}{l + 1} \int_{A_n(x)} f(t)^{p(x)} dt \right)^q \left( \frac{p(x)}{p(x)} \right) (l + 1)^q
\]

\[
\leq (c \ln 2 (l + 1)) q^*.
\]

\[
\cdot \int_0^l \frac{dx}{x} \left( \frac{1}{l + 1} \left[ \int_{A_n(x)} f(t)^{p(t)} + 1 \right] \right)^q \left( \frac{p(x)}{p(x)} \right) dx \leq 2^{-n\varepsilon q} C q^* (c \ln 2) q^* (l + 1)^{q^* - 1}.
\]
\[ \int_0^x \left( \int_{A_n(x)} [(f(t)^{p(t)} + 1)] \, dt \right) \, dx \]

Hence,
\[
I_q \left( x^{\alpha(x)} \int_{A_n(x)} f(t) \, dt \right) \leq c_3 2^{-nq} C q^+ \int_0^1 \left( f(x)^{p(t)} + 1 \right) \, dt \frac{dx}{x}
\]
\[
\leq c_3 \int_0^{2^{-nl}} \left( f(x)^{p(t)} + 1 \right) \frac{dx}{x}
\]
\[
= C q^+ c_3 2^{-nq} \ln 2 \int_0^{2^{-nl}} \left( f(x)^{p(t)} + 1 \right) \, dt
\]
\[
\leq c_4 q^+ c_3 2^{-nq} \ln 2 \left( 1 + 2^{-nl} \right) = c_4 2^{-nq}.
\]

Therefore, it has been proved that
\[
I_q \left( x^{\frac{1}{p^t}} \frac{1}{q} \int_{A_n(x)} f(t) \, dt \right) \leq c_4 2^{-nq},
\]
which implies
\[
\left\| x^{\frac{1}{p^t}} \frac{1}{q} \int_{A_n(x)} f(t) \, dt \right\|_{L_{p Murphy}} \leq c_4 q^+ 2^{-nq} q^-
\]
(6)

Inserting (6) in (3), we get
\[
\left\| x^{\frac{1}{p^t}} \frac{1}{q} \left( f(t) \right) \right\|_{L_{p Murphy}} \leq c_4 q^+ \sum_{n=1}^{\infty} 2^{-nq} q^-
\]

The Theorem A has been proved.

**Theorem 1.3.** Let
\[
q, p : (0, l) \rightarrow (1, \infty)
\]
be measurable functions such that
\[
\infty > q^+ \geq q(x) \geq q^- > p^+ \geq p(x) \geq p^- > 1
\]
Assume that the function \( p \) increases on \((0, l)\) and \( x^{\frac{1}{p^t} + \infty} \) is almost decreasing in \((0, l)\). Then operator \( H \) acts compactly the space \( L^{p^\prime, q^\prime}(0, l) \) into \( L^{q^-, p^-}(0, l) \) for any \( \delta \in (0, 1) \).

**Proof.** In order to prove Theorem 1.3, we may apply the approaches from [3,4,5]. In this way, insert the operators

\[ P_1f(x) = X_{(0, a)}(x)x^{-\frac{1}{p^t}} \int_0^x f(t) \, dt \]
\[ P_2f(x) = X_{(a, l)}(x)x^{-\frac{1}{p^t}} \int_0^a f(t) \, dt \]
\[ P_3f(x) = X_{(a, l)}(x)x^{-\frac{1}{p^t}} \int_a^x f(t) \, dt \]

As it was stated in [3], \( P_2 \) is a limit of finite rank operators, while \( P_3 \) is a finite rank operator. From the condition \( \lim_{l \to 0} B(t) = 0 \) it follows that
\[
\| Hf - P_2f - P_3f \|_{L^{q^\prime}(0, l)} \leq \| P_1f \|_{L^{q^\prime}(0, l)} \leq \delta \]
\[
\| H - P_2 - P_3 \|_{L^{p^\prime, q^\prime}(0, l)} \leq \| P_2 \|_{L^{p^\prime, q^\prime}(0, l)} \leq \delta \]
\[
\| H - P_2 \|_{L^{p^\prime, q^\prime}(0, l)} \leq \delta \]
\[
\leq ca^p \to 0, \ a \to 0.
\]
(7)

To show the last estimation we shall use the argu of Theorem 1.1. Repeating all constructions there, we get the following estimates
\[
I_q \left( x^{\frac{1}{p^t} \frac{1}{q^\prime} \frac{1}{q}} \int_{A_n(x)} f(t) \, dt \right) \]
\[
= \int_0^l \frac{dx}{x^{1-\delta+\epsilon}} \left( x^{\frac{1}{p^'+ \epsilon}} \right) \int_{A_n(x)} f(t) \, dt \]
\[
= C q^+ 2^{-nq} \int_0^l \frac{dx}{x^{1-\delta} X_{A_n(x)}(x)} \left( \int t^{-\frac{1}{p^t} - \infty} f(t) \, dt \right) \]

Notice, we have used \( x^{-\frac{1}{p^t} \frac{1}{q^\prime} \frac{1}{q}} \leq ct^{-\frac{1}{p^t} \frac{1}{q^\prime} \frac{1}{q}} \) for any \( t \in A_n(x) \).

Therefore, and using Holder’s inequality
\[
I_q \left( x^{\frac{1}{p^t} \frac{1}{q^\prime} \frac{1}{q}} \int_{A_n(x)} f(t) \, dt \right) \]
\[
\leq C q^+ 2^{-nq} \int_0^l \frac{dx}{x^{1-\delta} X_{A_n(x)}(x)} \left( \int (f(t)^{p^t \infty}) \, dt \right) \]

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Applying Lemma 1.2 and argues above, we attain the estimates

\[
I_q \left( x^{\frac{1}{p'} - \frac{1}{q}} \int_{A(n(x))} f(t) dt \right) \leq C q^+ 2^{-neq^-} (c \ln 2) q^+(l+1) q^{q^- - 1} \cdot \int_0^t \frac{dx}{x^{1-\delta}} \left( \int_{A(n(x))} [(f(t))^{p(t)} + 1] dt \right) \leq C_3 2^{-neq^-} \int_0^t \left( \int_{A(n(x))} dx \right) dt \leq 2^{-neq^-} (1 + 2^{-n} l) C_3 q^+ l^\delta
\]

Therefore, it has been shown that

\[
I_q \left( x^{\frac{1}{p'} - \frac{1}{q}} \int_{A(n(x))} f(t) dt \right) \leq cl^\delta 2^{-neq^-}
\]

if \( \|f\|_p \leq 1 \). This implies

\[
\left\| I_q \left( x^{\frac{1}{p'} - \frac{1}{q}} \int_{A(n(x))} f(t) dt \right) \right\|_{q:(0,l)} \leq C q^+ l^\delta 2^{-neq^-} q^{q^-}
\]

Inserting this estimates over \( n = 1, 2, \ldots \) in the expression

\[
\left\| x^{\frac{1}{p'} - \frac{1}{q}} Hf \right\|_{q:(0,l)} \leq c q^+ l^\delta \sum_{n=1}^{\infty} 2^{-n e q^-} q^+ = c_2 q^+ l^\delta
\]

The last estimate is a needed estimation which completes the proof of Theorem 1.3.

**Conclusion.** In this study, we obtained a new boundedness and compactness for the Hardy operator in variable exponential Lebesgue spaces (VELS), \( L^{p(.)}(0,l) \).

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