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Categories internal to crossed modules

Tunçar Şahan\textsuperscript{*1} and Jihad Jamil Mohammed\textsuperscript{2}

Abstract

In this study, internal categories in the category of crossed modules are characterized and it has been shown that there is a natural equivalence between the category of crossed modules over crossed modules, i.e. crossed squares and the category of internal categories within the category of crossed modules. Finally, we obtain examples of crossed squares using this equivalence.

Keywords: Crossed module, internal category, crossed square

1. INTRODUCTION

Crossed modules are first defined in the works of Whitehead [25-27] and has been found important in many areas of mathematics including homotopy theory, group representation theory, homology and cohomology on groups, algebraic K-theory, cyclic homology, combinatorial group theory and differential geometry. See [4-7] for applications of crossed modules. Later, it was shown that the categories of internal categories in the category of groups and the category of crossed modules are equivalent [8-14].

Mucuk et al. [18] interpret the concept of normal subcrossed module and quotient crossed module concepts in the category of internal categories within groups, that is group-groupoids. The equivalences of the categories given in [8, Theorem 1] and [24, Section 3] enable us to generalize some results on group-groupoids to the more general internal groupoids for an arbitrary category of groups with operations (see for example [1], [15], [16] and [17]).


Crossed squares are first described to be applied to algebraic K-theoretic problems [12]. Crossed squares are two-dimensional analogous of crossed modules and model all connected homotopy 3-types (hence all 3-groups) and correspond in much the same way to pairs of normal subgroups while crossed modules model all connected homotopy 2-types and groups model all connected homotopy 1-types.

Recently, freeness conditions for 2-crossed modules and crossed squares are given in [19] and [20]. See also [3] for commutative algebra case.

Main objective of this study is to characterize internal categories within the category of crossed modules and to prove that the category of internal categories in the category of crossed modules and the category of crossed squares are equivalent. Hence this equivalence allow us to produce more examples of crossed squares.

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2. PRELIMINARIES

In this section we recall some well-known basic definitions and results.

2.1. Extensions and crossed modules

Following are detailed descriptions of the ideas given in [24] for the case of groups. An exact sequence of the form

\[ 0 \rightarrow A \xrightarrow{i} E \xrightarrow{p} B \rightarrow 0 \]

is called a short exact sequence where 0 is the group with one element. Here \(i\) is a monomorphism, \(p\) is an epimorphism and \(\ker p = A\). In a short exact sequence, group \(E\) is called an extension of \(B\) by \(A\). An extension is called split if there exist a group homomorphism \(s: B \rightarrow E\) such that \(ps = 1_B\).

Let \(E\) be a split extension of \(B\) by \(A\). Then the function

\[ \theta : E \rightarrow A \times B, \quad e \mapsto (e - sp(e), p(e)) \]

is a bijection. The inverse of \(\theta\) is given by \(\theta^{-1}(a, b) = a + s(b)\).

Thus it is possible to define a group structure on \(A \times B\) such that \(\theta\) is an isomorphism of groups. Let \((a, b), (a_i, b_i) \in A \times B\). Then

\[(a, b) + (a_i, b_i) = \theta^{-1}(\theta^{-1}(a, b) + \theta^{-1}(a_i, b_i))
= \theta(a + s(b) + a_i + s(b_i))
= (a + s(b) + a_i + s(b_i) - s(b_i) - s(b), b + b_i)
= (a + (s(b) + a_i - s(b)), b + b_i).\]

Here we note that a split extension of \(B\) by \(A\) defines an (left) action of \(B\) on \(A\) with

\[ b \cdot a = s(b) + a - s(b) \]

for \(a \in A\) and \(b \in B\). \(A \times B\) is called the semi-direct product group of \(A\) and \(B\) with the operation given above and denoted by \(A \rtimes B\).

These kind of actions are called derived actions [23]. Every group \(A\) has a split extension by itself in a natural way which gives rise to the conjugation action as

\[ 0 \rightarrow A \xrightarrow{i} A \rtimes A \xrightarrow{\rho} A \rightarrow 0 \]

where \(i(a) = (a, 0)\), \(p(a, a_i) = a_i\) and \(s(a) = (0, a)\) for \(a, a_i \in A\).

**Definition 2.1** Let \(A\) and \(B\) be two groups and let \(B\) acts on \(A\) on the left. Then a group homomorphism \(\alpha : A \rightarrow B\) is called a crossed module if \(1 \times \alpha : A \rightarrow A \rtimes A\) and \(\alpha \times 1_B : A \rtimes B \rightarrow B\) are group homomorphisms [24].

A crossed module is denoted by \((A, B, \alpha)\). It is useful to give the definition of crossed modules in terms of group operations and actions.

**Proposition 2.2** Let \(A\) and \(B\) be two groups, \(\alpha : A \rightarrow B\) a group homomorphism and \(B\) acts on \(A\) on the left. Then \((A, B, \alpha)\) is a crossed module if and only if

1. \((CM1)\) \(\alpha(b \cdot a) = b + \alpha(a) - b\)
2. \((CM2)\) \(\alpha(a) \cdot a_i = a + a_i - a\)

for all \(a, a_i \in A\) and \(b \in B\) [24].

**Example 2.3** Following homomorphisms are standard examples of crossed modules.

(i) Let \(X\) be a topological space, \(A \subset X\) and \(x \in A\). Then the boundary map \(\rho\) from the second relative homotopy group \(\pi_2(X, A, x)\) to the fundamental group \(\pi_1(X, x)\) is a crossed module with the natural action given in [27].

(ii) Let \(G\) be a group and \(N\) a normal subgroup of \(G\). Then the inclusion function \(N \xrightarrow{inc} G\) is a crossed module where the action of \(G\) on \(N\) is conjugation.

(iii) Let \(G\) be a group. Then the inner automorphism map \(G \rightarrow \text{Aut}(G)\) is a crossed module. Here the action is given by \(\psi \cdot g = \psi(g)\) for all \(\psi \in \text{Aut}(G)\) and \(g \in G\).

(iv) Given any \(G\)-module, \(M\), the trivial homomorphism \(0 : M \rightarrow G\) is a crossed \(G\)-module with the given action of \(G\) on \(M\).

A morphism \(f = \langle f_A, f_B \rangle\) of crossed modules from \((A, B, \alpha)\) to \((A', B', \alpha')\) is a pair of group
homomorphisms $f_A : A \rightarrow A'$ and $f_B : B \rightarrow B'$ such that $f_a \alpha = \alpha' f_a$ and $f_A (b \cdot a) = f_B (b) \cdot f_A (a)$ for all $a \in A$ and $b \in B$.

Crossed modules form a category with morphisms defined above. The category of crossed modules is denoted by $\text{XMod}$.

**Definition 2.4** Let $(A, B, \alpha)$ and $(S, T, \sigma)$ be two crossed modules. Then $(S, T, \sigma)$ is called a subcrossed module of $(A, B, \alpha)$ if $S \leq A$, $T \leq B$, $\sigma$ is the restriction of $\alpha$ to $S$ and the action of $T$ on $S$ is the induced action from that of $B$ on $A$ [21,22].

**Definition 2.5** Let $(A, B, \alpha)$ be a crossed module and $(S, T, \sigma)$ a subcrossed module of $(A, B, \alpha)$. Then $(S, T, \sigma)$ is called a normal subcrossed module or an ideal of $(A, B, \alpha)$ if

(i) $T \leq B$,
(ii) $b \cdot s \in S$ for all $b \in B$, $s \in S$ and
(iii) $t \cdot a - a \in S$ for all $t, a \in A$ [21,22].

**Example 2.6** Let $f : (A, B, \alpha) \rightarrow (A', B', \alpha')$ be a morphism of crossed modules. Then the kernel $\ker f = \ker \left(f_A, f_B \right) = \left(\ker f_A, \ker f_B, \alpha_{ker f_A} \right)$ of $f = \left(f_A, f_B \right)$ is a normal subcrossed module (ideal) of $(A, B, \alpha)$.

Moreover, the image $\text{Im} f = \text{Im} \left(f_A, f_B \right) = \left(\text{Im} f_A, \text{Im} f_B, \alpha'_{\text{Im} f_A} \right)$ of $f = \left(f_A, f_B \right)$ is a subcrossed module of $(A', B', \alpha')$.

**Definition 2.7** A topological crossed module $(A, B, \alpha)$ is a crossed module where $A$ and $B$ are topological groups such that the boundary homomorphism $\alpha : A \rightarrow B$ and the action of $B$ on $A$ are continuous.

Now we give the pullback notion in the category of crossed modules.

**Definition 2.8** Let $(A, B, \alpha)$, $(M, P, \mu)$ and $(C, D, \gamma)$ be three crossed modules and $f = \left(f_A, f_B \right) : (A, B, \alpha) \rightarrow (M, P, \mu)$ and $g = \left(g_C, g_D \right) : (C, D, \gamma) \rightarrow (M, P, \mu)$ be two crossed module morphisms. Then the pullback crossed module of $f$ and $g$ is $\left(A_{g_{12}}, B_{g_{12}}, D_{\alpha \times \gamma}, C_{f_{12}}, B_{f_{12}}, D_{\alpha \times \gamma} \right)$ where the action of $B_{f_{12}} \times D_{\alpha \times \gamma}$ on $A_{g_{12}} \times C_{f_{12}}$ is given by

$$(b, d) \cdot (a, c) = (b \cdot a, d \cdot c)$$

for all $(b, d) \in B_{f_{12}} \times D_{\alpha \times \gamma}$ and $(a, c) \in A_{g_{12}} \times C_{f_{12}}$.

2.2. **Internal categories and Brown-Spencer Theorem**

**Definition 2.9** Let $\mathfrak{C}$ be a category with pullbacks. Then an internal category $C$ in $\mathfrak{C}$ consist of two objects $C_1$ and $C_0$ in $\mathfrak{C}$ and four structure morphisms $s, t : C_1 \rightarrow C_0$, $e : C_0 \rightarrow C_1$ and $m : C_1 \times C_1 \rightarrow C_1$, where $C_1 \times C_1$ is the pullback of $s$ and $t$, such that the following conditions hold:

(i) $s e = t e = 1_{C_0}$;
(ii) $s m = s \pi_2$, $t m = t \pi_1$;
(iii) $m \left(1_{C_0}, m \right) = m \left(m \times 1_{C_0} \right)$ and
(iv) $m \left(e s, 1_{C_1} \right) = m \left(1_{C_1}, e t \right) = 1_{C_1}$.

Morphisms $s, t, e$ and $m$ are called source, target, identity object maps and composition respectively. An internal category in $\mathfrak{C}$ will be denoted by $C = (C_1, C_0, s, t, e, m)$ or only by $C$ for short.

If there is a morphism $n : C_1 \rightarrow C_1$ in $\mathfrak{C}$ such that $m(1_{C_0}, n) = e s$ and $m(n, 1) = e t$, i.e. every morphism in $C_1$ has an inverse up to the composition, then we say that $C = (C_1, C_0, s, t, e, n, m)$ is an internal groupoid in $\mathfrak{C}$.

Let $C$ and $C'$ be two internal categories in $\mathfrak{C}$. Then a morphism $f = (f_1, f_0)$ from $C$ to $C'$ consist of a pair of morphisms $f_1 : C_1 \rightarrow C'_1$ and $f_0 : C_0 \rightarrow C'_0$ in $\mathfrak{C}$ such that

(i) $s f_1 = f_0 s$, $t f_1 = f_0 t$,
(ii) $e f_0 = f_1 e$ and
(iii) $m(f_1 \times f_0) = f_1 m$.

Thus one can construct the category of internal categories in an arbitrary category $\mathfrak{C}$ with pullbacks where the morphisms are morphisms of internal
categories as given above. This category is denoted by $\text{Cat}(\mathcal{S})$.

An internal category in the category of groups is called a group-groupoid [8]. Group-groupoids are also the group objects in the category of small categories.

**Example 2.10** Let $X$ be a topological group. Then the set $\pi X$ of all homotopy classes of paths in $X$ defines a groupoid structure on the set of objects $X$. This groupoid is called the fundamental groupoid of $X$. Moreover, $\pi X$ is a groupoid [8].

Let $G$ be an internal category in the category of groups, i.e. a group-groupoid. Then the object of morphisms $G_i$ and object of objects $G_0$ have group structures and there are four group homomorphisms $s,t:G_i \rightarrow G_0$, $e:G_0 \rightarrow G_i$ and $m:G_i \times G_i \rightarrow G_i$ such that the conditions (i)-(iv) of Definition 2.9 are satisfied.

Morphisms between group-groupoids are functors which are group homomorphisms. The category of group-groupoids is denoted by $\text{GpGd}$.

Since $m:G_i \times G_i \rightarrow G_i$ is a group homomorphism then we can give the following lemma.

**Lemma 2.11** Let $G$ be an internal category in the category of groups. Then

$$m((b,a)+(b',a'))=m((b',a'))+m((b',a')),$$

i.e.

$$(b+b')(a+a')=(b\circ a)+(b'\circ a')$$

whenever one side (hence both sides) make senses, for all $a,a',b,b' \in G_i$ [8].

Equation given in Lemma 2.11 is called the interchange law. Applications of interchange law can be given as in the following.

Let $G$ be a group-groupoid. Then the partial composition in $G$ can be given in terms of group operations [8]. Indeed, let $a \in G(x,y)$ and $b \in G(y,z)$. Then

$$b\circ a = (b+0)\circ (1_z + (-1_x + a))$$
$$= (b\circ 1_y) + (0 \circ (-1_x + a))$$
$$= b - 1_y + a$$

and similarly $b\circ a = a - 1_y + b$.

**Corollary 2.12** Let $G$ be a group-groupoid. Then the elements of $\ker s$ and $\ker t$ commute under the group operation [8].

One can give the inverse of a morphism in terms of group operation as another consequence of the interchange law. That is, let $a \in G(x,y)$. Then

$$1_y = a \circ a^{-1} = a - 1_y + a^{-1}.$$  

Thus $a^{-1} = 1_y - a + 1_y$. Similarly $a^{-1} = 1_y - a + 1_y$.

A final remark is that if $a,a \in \ker s$ and $t(a) = x$ then $-1_x + a \in \ker t$ so commutes with $a$. This implies that

$$( -1_x + a) + a = a + (-1_x + a)$$

and thus

$$a - a = 1_x + a - 1_y.$$  

Let $G$ be an internal category in the category of groups and $G$ be a group-groupoid. Then $\pi X$ is a group-groupoid [8].

**Theorem 2.13** [Brown & Spencer Theorem] The category $\text{GpGd}$ of group-groupoids and the category $\text{XMod}$ of crossed modules are equivalent [8].

**Proof:** We sketch the proof since we need some details in the last section. Define a functor

$$\varphi: \text{GpGd} \rightarrow \text{XMod}$$

as follows: Let $G$ be a group-groupoid. Then $\varphi(G) = (A,B,\alpha)$ is a crossed modules where $A = \ker s$, $B = G_0$, $\alpha$ is the restriction of $t$ and the action of $B$ on $A$ is given by $x \cdot a = 1_x + a - 1_x$.

Conversely, define a functor

$$\psi: \text{XMod} \rightarrow \text{GpGd}$$

as follows: Let $(A,B,\alpha)$ be a crossed module. Then the semi-direct product group $A \alpha B$ is a group-groupoid on $B$ where $s(a,b) = b$, $t(a,b) = \alpha(a) + b$, $e(b) = (0,b)$ and the composition is

$$(a',b') \circ (a,b) = (a' + a, b)$$

where $b' = \alpha(a) + b$. Other details are straightforward so is omitted.
3. INTERNAL CATEGORIES WITHIN THE CATEGORY OF CROSSED MODULES

In this section we will characterize internal categories in the category XMod. Let C be an internal category in the category XMod of crossed modules over groups. Then C consist of two crossed modules $C_1 = (A_1, B_1, \alpha_1)$ and $C_0 = (A_0, B_0, \alpha_0)$ and four crossed module morphisms as $s = (s_A, s_B)$, $t = (t_A, t_B)$: $C_1 \rightarrow C_0$ which are called the source and the target maps respectively, $\varepsilon = (\varepsilon_A, \varepsilon_B)$: $C_0 \rightarrow C_1$ which is called the identity object map and $m = (m_A, m_B)$: $C_1 \times C_1 \rightarrow C_1$ which is called the composition map. These are object to the followings:

(i) $s \varepsilon = t \varepsilon = 1_{C_1}$;

(ii) $s m = s \pi_2$, $t m = t \pi_1$;

(iii) $m(1_{C_1} \times m) = m(m \times 1_{C_1})$ and

(iv) $m(\varepsilon_A, 1_{C_1}) = m(1_{C_1}, \varepsilon_T) = 1_{C_1}$.

An internal category in the category XMod will be denoted by $C = (C_1, C_0, s, t, \varepsilon, m)$ or briefly by $C$ when no confusion arise. Identity objects $\varepsilon_A(a_0)$ and $\varepsilon_B(b_0)$ will be denoted by $1_{b_0}$ and $1_{a_0}$ for short, respectively. Also the composition of elements will be denoted by $m_A(a_i, a_i') = a_i \circ a_i'$ and by $m_B(b_i, b_i') = b_i \circ b_i'$ for $a_i, a_i' \in A_i$ and $b_i, b_i' \in B_i$ with $s_A(a_i) = t_A(a_i')$ and $s_B(b_i) = t_B(b_i')$.

Example 3.1 Let $(A, B, \alpha)$ be a crossed module over groups. We know that $(A \times A, B \times B, \alpha \times \alpha)$ is also a crossed module. If we set $C_i = (A \times A, B \times B, \alpha \times \alpha)$, $C_0 = (A, B, \alpha)$, $s = \pi_1$, $t = \pi_2$, $\varepsilon = \Delta$ and define $m$ with $(a_i, a_i') \circ (a_i, a_i') = (a_i, a_i)$ and $(b_i, b_i') \circ (b_i, b_i') = (b_i, b_i)$ for all $a_i, a_i' \in A$ and $b_i, b_i' \in B$ then $C = (C_i, C_0, s, t, \varepsilon, m)$ becomes an internal category in XMod.

Example 3.2 Let $(A, B, \alpha)$ be a crossed module over groups. Then $C = ((A, B, \alpha), (A, B, \alpha), s, t, \varepsilon, m)$ becomes an internal category in XMod where $s$, $t$ and $\varepsilon$ are identity maps.

Example 3.3 Let $(A, B, \alpha)$ be a topological crossed module. Then $(\pi A, \pi B, \pi \alpha)$ is also a crossed module. Moreover,

$\pi(A, B, \alpha) = ((\pi A, \pi B, \pi \alpha), (A, B, \alpha), s, t, \varepsilon, m)$

is an internal category in XMod.

Now we will give the properties of an internal category with a few lemmas individually.

Lemma 3.4 Let $C$ be an internal category in XMod. Then for $i \in \{0, 1\}$

(i) $\alpha_i(a_i + a_i') = \alpha_i(a_i) + \alpha_i(a_i')$,

(ii) $\alpha_i(b_i \cdot a_i) = b_i + \alpha_i(a_i) - b_i$ and

(iii) $\alpha_i(a_i) \cdot a_i' = a_i + a_i' - a_i$

for all $a_i, a_i' \in A_i$ and $b_i \in B_i$.

Proof: It follows from the fact that $C_i = (A_i, B_i, \alpha_i)$ is a crossed module for $i \in \{0, 1\}$.

Lemma 3.5 Let $C$ be an internal category in XMod. Then

(i) $s_A(a_i + a_i') = s_A(a_i) + s_A(a_i')$,

(ii) $b_i + b_i' = s_B(b_i) + s_B(b_i')$,

(iii) $t_A(a_i + a_i') = t_A(a_i) + t_A(a_i')$,

(iv) $b_i + b_i' = t_B(b_i) + t_B(b_i')$,

(v) $a_0 s_A = s_B a_1$, $a_0 t_A = t_B a_1$. 

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This means \( a_i \circ a_i' = a_i - 1_{s_i(a_i)} + a_i' = a_i - 1_{s_i(a_i)} + a_i \) and
\[
b_i \circ b_i' = b_i - 1_{s_i(b_i)} + b_i' = b_i - 1_{s_i(b_i)} + b_i
\]
for \( a_i, a_i' \in A \), \( b_i, b_i' \in B \) with \( s_A(a_i) = t_A(a_i) \) and \( s_B(b_i) = t_B(b_i) \).

**Proof:** We will prove the assumption for \( A \). If \( 0 \) denotes the identity (zero) elements of groups \( A \) and \( B \) then
\[
a_i \circ a_i' = (a_i + 0) \circ (1_{s_i(a_i)} + (1_{s_i(a_i)} + a_i')) = (a_i \circ 1_{s_i(a_i)}) + 0 \circ (1_{s_i(a_i)} + a_i') = a_i - 1_{s_i(a_i)} + a_i'
\]
and similarly
\[
a_i \circ a_i' = (0 + a_i) \circ (1_{s_i(a_i)} + 1_{s_i(a_i)}) = 0 \circ (1_{s_i(a_i)} + 1_{s_i(a_i)}) + (a_i \circ 1_{s_i(a_i)}) = a_i - 1_{s_i(a_i)} + a_i.
\]

By this corollary we obtain that if \( s_A(a_i) = t_A(a_i') = 0 \), i.e. \( a_i \in \ker s_A \) and \( a_i' \in \ker t_A \), then
\[
a_i + a_i' = a_i' + a_i.
\]

So the elements of \( \ker s_A \) and \( \ker t_A \) are commutative. Similarly, the elements of \( \ker s_B \) and \( \ker t_B \) are commutative too. Moreover, for an element \( a_i \in A \), \( a_i^{-1} = 1_{t_i(a_i)} - a_i + 1_{s_i(a_i)} \in A \) is the inverse element of \( a_i \), up to the composition \( m_i \). Similarly for an element \( b_i \in B \), \( b_i^{-1} = 1_{s_i(b_i)} - b_i + 1_{t_i(b_i)} \in B \) is the inverse element of \( b_i \) up to the composition \( m_i \). This means that \( C = \langle C_i, C_0, s, t, e, m, n \rangle \) has a groupoid structure where \( n = \langle n_i, n_B \rangle \): \( C_i \to C_1 \) is a morphism of crossed modules where
\[
n_A : A_i \to A_i \\
a_i \mapsto n_A(a_i) = a_i^{-1} = 1_{s_i(a_i)} - a_i + 1_{s_i(a_i)}
\]
and
\[ n_\beta : B_1 \to B_0 \quad \quad b_1 \mapsto n_\beta(b_1) = b_1^{-1} = 1_{s_0}^{-1} - b_1^{-1} + 1_{s_1}^{-1}. \]

It is easy to see that

\[ 1_{s_1(a)}^{-1} - a_1 + 1_{s_0(a)}^{-1} = 1_{s_1(a)} - a_1 + 1_{s_0(a)} \]

for all \( a_1 \in A_1 \) and similarly

\[ 1_{s_1(b)}^{-1} - b_1 + 1_{s_0(b)}^{-1} = 1_{s_1(b)}^{-1} - b_1 + 1_{s_0(b)}^{-1} \]

for all \( b_1 \in B_1 \).

**Lemma 3.7** Let \( a_1 \in A_1 \) and \( b_1 \in B_1 \). Then

\[ b_1^{-1} \cdot a_1^{-1} = \left( b_1 \cdot a_1 \right)^{-1}. \]

**Proof:** By the condition (ix) of Lemma 3.5

\[
\left( b_1 \cdot a_1 \right) \circ \left( b_1^{-1} \cdot a_1^{-1} \right) = \left( b_1 \circ b_1^{-1} \right) \circ \left( a_1 \circ a_1^{-1} \right)
\]

\[
= 1_{s_1(b_1)}^{-1} \cdot 1_{s_0(a_1)}^{-1}
\]

\[
= 1_{s_1(b_1)} \cdot 1_{s_0(a_1)}
\]

\[
= 1_{s_1(b_1 \cdot a_1)}
\]

and similarly \( \left( b_1^{-1} \cdot a_1^{-1} \right) \circ \left( b_1 \cdot a_1 \right) = 1_{s_1(b_1 \cdot a_1)} \). Thus

\[ b_1^{-1} \cdot a_1^{-1} = \left( b_1 \cdot a_1 \right)^{-1}. \]

It is easy to see that an internal category in the category of crossed modules over groups is indeed a crossed module object in the category of internal categories within groups.

**Definition 3.8** Let \( C \) and \( C' \) be two internal categories in \( \text{XMod} \). A morphism (internal functor) from \( C \) to \( C' \) is a pair of crossed module morphisms

\[ f = \left( f_1 = \left( f_1^A, f_1^B \right), f_0 = \left( f_0^A, f_0^B \right) \right) : C \to C' \]

such that

\[ f_0 \delta = s f_1, \quad f_0 t = t f_1, \quad f_0 \epsilon = e f_0 \]

and

\[ f m = m f_1. \]

Hence we can construct the category of internal categories (groupoids) within the category of crossed modules over groups where the morphisms are internal functors as defined above. This category will be denoted by \( \text{Cat(Xmod)} \).

### 3.1. Crossed squares

Crossed squares are first defined in [12]. In this subsection we recall the definition of a crossed square as given in [7]. Further we prove that the category of crossed squares and that of internal categories within crossed modules are equivalent. Finally we give some examples of crossed squares using this equivalence.

**Definition 3.9** A crossed square over groups consists of four morphisms of groups \( \lambda : L \to M \), \( \lambda' : L \to N \), \( \mu : M \to P \) and \( \nu : N \to P \) together with actions of the group \( P \) on \( L, M, N \) on the left, conventionally, and hence actions of \( M \) on \( L \) and \( N \) via \( \mu \) and of \( N \) on \( L \) and \( M \) via \( \nu \) and a function \( h : M \times N \to L \). These are subject to the following axioms:

(i) \( \lambda, \lambda' \) are \( P \)-equivariant and \( \mu, \nu \) and

\[ \kappa = \mu \lambda = \nu \lambda' \] are crossed modules,

(ii) \( \lambda h(m,m') = m + n \cdot ( - m) \), \( \lambda'h(m,m) = m \cdot n - n \),

(iii) \( h(\lambda(l),n) = l + n \cdot ( - l) \), \( h(m,\lambda'(l)) = m \cdot l - l \),

(iv) \( h(m + m',n) = m \cdot h(m',n) + h(m,n) \), \( h(m,n + n') = h(m,n) + n \cdot h(m,n') \),

(v) \( h(p \cdot m, p \cdot n) = p \cdot h(m,n) \)

for all \( l \in L \), \( m, m' \in M \), \( n', n' \in N \) and \( p \in P \) [7].

A crossed square will be denoted by \( S = (L, M, N, P) \).

**Example 3.10.** Let \((A, B, \sigma)\) be crossed module and \((S, T, \sigma)\) a normal subcrossed module of \((A, B, \alpha)\). Then

\[
\begin{array}{c}
\xymatrix{L \ar[r]^{\lambda} & M} \\
N \ar[u]^{\lambda'} \ar[r]^{\mu} & P \ar[u]_{\nu}
\end{array}
\]

forms a crossed square of groups where the action of \( B \) on \( S \) is induced action from the action of \( B \) on \( A \) and the action of \( B \) on \( T \) is conjugation. The h map is defined by \( h(t,a) = t \cdot a - a \) for all \( t \in T \) and \( a \in A \) [21,22].

A topological example of crossed squares is the fundamental crossed square which is defined in [7] as follows: Suppose given a commutative square of spaces.
Let $F(f)$ be the homotopy fibre of $f$ and $F(X)$ the homotopy fibre of $F(g) \to F(a)$. Then the commutative square of groups

\[
\begin{array}{ccc}
\pi_1 F(X) & \longrightarrow & \pi_1 F(g) \\
\downarrow & & \downarrow \\
\pi_1 F(f) & \longrightarrow & \pi_1 (C)
\end{array}
\]

is naturally equipped with a structure of crossed square. This crossed square is called the fundamental crossed square [7].

A morphism $f = (f_L, f_M, f_N, f_P)$ of crossed squares from $S_1 = (L_1, M_1, N_1, P_1)$ to $S_2 = (L_2, M_2, N_2, P_2)$ consist of four group homomorphisms $f_L : L_1 \to L_2$, $f_M : M_1 \to M_2$, $f_N : N_1 \to N_2$ and $f_P : P_1 \to P_2$ which are compatible with the actions and the functions $h_1$ and $h_2$ such that the following diagram is commutative.

\[
\begin{array}{ccc}
\begin{array}{c}
\lambda_1 \\
M_1
\end{array} & \xrightarrow{f_M} & \begin{array}{c}
\lambda_2 \\
M_2
\end{array} \\
\downarrow f_L & & \downarrow f_M \\
L_1 & \xrightarrow{\mu_2} & L_2
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
\lambda'_1 \\
N_1
\end{array} & \xrightarrow{f_P} & \begin{array}{c}
\lambda'_2 \\
N_2
\end{array} \\
\downarrow f_N & & \downarrow f_P \\
\mu_1 & & \mu_2
\end{array}
\]

Category of crossed squares over groups with morphisms between crossed squares defined above is denoted by $\mathbf{X^3Mod}$. Crossed squares are equivalent to crossed modules over crossed modules [22].

Now we prove our main theorem.

**Theorem 3.11.** The category $\mathbf{Cat(XMod)}$ of internal categories within the category of crossed modules over groups and the category $\mathbf{X^3Mod}$ of crossed squares over groups are equivalent.

**Proof:** We first define a functor $\eta : \mathbf{Cat(XMod)} \to \mathbf{X^3Mod}$ as follows: Let $C = (C_1, C_0, s, t, e, m, n)$ be an object in $\mathbf{Cat(XMod)}$. If we set $L = \ker s_A$, $M = \ker s_B$, $N = A_0$, $P = B_0$,

\[
\lambda = \alpha_{\ker s_A}, \quad \lambda' = t_{\ker s_B}, \quad \mu = t_{\ker s_A} \quad \text{and} \quad \nu = \alpha_0
\]

then $\eta(C) = (L, M, \lambda, N, P)$ becomes a crossed square with the function $h(m, n) = m \cdot 1_n - 1_m$ for all $m \in M$ and $n \in N$.

Here $(L, M, \lambda)$ is a crossed module since it is the kernel crossed module of

\[
\eta(C) = (L, M, N, P)
\]

Moreover we know that $(L, N, \lambda')$ and $(M, P, \mu)$ are crossed modules by Brown & Spencer Theorem [8, Theorem 1]. Finally $(N, P, \nu)$ is already a crossed module since it is $\eta(C)$. Here the actions of $P$ on $N$ is already given, on $M$ is given by $p \cdot m = 1_p + m - 1_p$ and on $L$ is given by $p \cdot l = 1_p \cdot l$ (where the action on the right side of the equation is the action of $B$ on $A$) for $p \in P$, $m \in M$ and $l \in L$. Now we need to show that the conditions given in the Definition 3.9 is satisfied.

(i) We need to show that $\lambda$, $\lambda'$ are $P$-equivariant and $\kappa = \mu \lambda$ is a crossed module. Let $l \in L$, and $p \in P$. Then

\[
\lambda(p \cdot l) = \alpha_l(1_p \cdot l) = 1_p + \alpha_l(l) - 1_p = p \cdot \lambda(l)
\]

and

\[
\lambda'(p \cdot l) = t_A(1_p \cdot l) = t_B(1_p) \cdot t_A(l) = p \cdot \lambda'(l)
\]

so $\lambda$ and $\lambda'$ are $P$-equivariant. Now we need to show that $(L, P, \kappa)$ is a crossed module. So

(CM1) Let $l \in L$, and $p \in P$. Then

\[
\kappa(p \cdot l) = \mu \lambda(p \cdot l)
\]

\[
= \mu(p \cdot \lambda(l))
\]

\[
= p + \mu(\lambda(l)) - p
\]

\[
= p + \kappa(l) - p
\]
(CM2) Let \( l, l' \in L \). Then
\[
\kappa(l) \cdot l' = \mu(\lambda(l)) \cdot l'
\]
\[
= 1_{\mu(\lambda(l))} \cdot l'
\]
\[
= 1_{\lambda(\lambda(l))} \cdot l'
\]
\[
= \lambda(1_{\lambda(l)}) \cdot l'
\]
\[
= 1_{\lambda(l)} + l' - 1_{\lambda(l)}
\]
\[
= l + l' - l
\]

(ii) Let \( m \in M \) and \( n \in N \). Then
\[
\lambda h(m, n) = \lambda(m \cdot 1_n - 1_n)
\]
\[
= \lambda(m \cdot 1_n) - \lambda(1_n)
\]
\[
= m \cdot \lambda(1_n) - m - \lambda(1_n)
\]
\[
= m + n \cdot (-m)
\]
and
\[
\lambda' h(m, n) = \lambda'(m \cdot 1_n - 1_n)
\]
\[
= \lambda'(m \cdot 1_n) - \lambda'(1_n)
\]
\[
= \mu(m) \cdot \lambda'(1_n) - \lambda'(1_n)
\]
\[
= \mu(m) \cdot n - n
\]
\[
= m + n
\]

(iii) Let \( l \in L \), \( m \in M \) and \( n \in N \). Then
\[
h(\lambda(l), n) = \lambda(l) \cdot 1_n - 1_n
\]
\[
= (l + 1_n - l) - 1_n
\]
\[
= l + (1_n - l - 1_n)
\]
\[
= l + n \cdot (-l)
\]
and
\[
h(m, \lambda'(l)) = m \cdot 1_{\lambda'(l)} - 1_{\lambda(l)}
\]
\[
= (m \cdot 1_{\lambda'(l)} - 1_{\lambda(l)}) + l - l
\]
\[
= (m \cdot 1_{\lambda'(l)}) - l
\]
\[
= \left( m \cdot 1_{\lambda'(l)} \circ (1_{\lambda(l)} \circ l) \right) - l
\]
\[
= (m \cdot 1_{\lambda'(l)}) - l
\]
\[
= m \cdot l - l
\]

(iv) Let \( m, m' \in M \) and \( n, n' \in N \). Then
\[
h(m + m', n) = (m + m') \cdot 1_n - 1_n
\]
\[
= m \cdot (m' \cdot 1_n) - 1_n
\]
\[
= m \cdot (m' \cdot 1_n) + m \cdot (-1_n + 1_n) - 1_n
\]
\[
= m \cdot (m' \cdot 1_n - 1_n) + m \cdot 1_n - 1_n
\]
\[
= m \cdot h(m', n) + h(m, n)
\]
and
\[
h(m, n + n') = m \cdot 1_{n + n'} - 1_{n + n'}
\]
\[
= m \cdot (1_n + 1_{n'}) - 1_n - 1_{n'}
\]
\[
= (m \cdot 1_n - 1_n) + 1_n + (m \cdot 1_{n'} - 1_{n'}) - 1_n - 1_{n'}
\]
\[
= h(m, n) + n \cdot h(m, n')
\]

(v) Let \( m \in M \), \( n \in N \) and \( p \in P \). Then
\[
h(p \cdot m, p \cdot n) = h(1_p + m - 1_p, p \cdot n)
\]
\[
= (1_p + m - 1_p) \cdot 1_p - 1_p
\]
\[
= (1_p + m - 1_p) \cdot (1_p \cdot 1_n) - (1_p \cdot 1_n)
\]
\[
= 1_p \cdot (m \cdot 1_n) + 1_p \cdot (-1_n)
\]
\[
= 1_p \cdot (m \cdot 1_n - 1_n)
\]
\[
= p \cdot h(m, n)
\]

Now let
\[
f = (f_1, f_1^a, f_0^a, f_0^a) : C \rightarrow C'
\]

be a morphism in \( \text{Cat}(\text{XMod}) \). Then
\[
\eta(f) = (f_{1\text{ker}x}, f_{1\text{ker}x}^a, f_0^a, f_0^a) : S \rightarrow S'
\]
is a morphism of crossed squares.

Conversely define a functor
\[
\psi : X^2\text{Mod} \rightarrow \text{Cat}(\text{XMod})
\]
as follows: Let \( S = (L, M, N, P) \) be a crossed square over groups. Then
\[
\psi(S) = (C_1 = (A_1, B_1, \alpha_1), C_0 = (A_0, B_0, \alpha_0), s, t, \alpha, m)
\]
is an internal category within the category of crossed modules over groups where
\[
(A_1, B_1, \alpha_1) = (L \triangleleft N, M \triangleleft P, \lambda \times \nu),
\]
\[
(A_0, B_0, \alpha_0) = (N, P, \nu),
\]
\[
s_\lambda (l, n) = n, s_\gamma (m, p) = p,
\]
\[ t_\lambda(l,n) = \lambda'(l) + n, \quad t_\mu(m,p) = \mu(m) + p, \]
\[ \varepsilon_\lambda(n) = (0,n), \quad \varepsilon_\mu(p) = (0,p), \]
\[ (l',\lambda'(l) + n) \circ (l,n) = (l' + l,n) \]
and
\[ (m',\lambda'(m) + p) \circ (m,p) = (m' + m, p). \]

We know that \( C_0 \) is a crossed module over groups. First we need to show that \((L \Lambda N, M \Lambda P, \lambda \times \nu)\) is a crossed module with the action of \( M \Lambda P \) on \( L \Lambda N \) is
\[ (m,p) \cdot (l,n) = (m \cdot (p \cdot l) + h(m,p \cdot n), p \cdot n). \]

**CM1** Let \((l,n) \in L \Lambda N \) and \((m,p) \in M \Lambda P \).

Then
\[ (\lambda \times \nu)((m,p) \cdot (l,n)) = (\lambda \times \nu)(m \cdot (p \cdot l) + h(m,p \cdot n), p \cdot n) = (m,p) + (\lambda \times \nu)(l,n) - (m,p). \]

**CM2** Let \((l,n),(l',n') \in L \Lambda N \). Then
\[ (\lambda \times \nu)((l,n)) \cdot (l',n') = ((\lambda(l), \nu(n)) \cdot (l',n') = (l,n) - (l',n') + (l',n') - (l,n). \]

Thus \( C = (L \Lambda N, M \Lambda P, \lambda \times \nu) \) is a crossed module.

Now we need to show that \( \varphi(S) = C \) satisfies the conditions given in Lemma 3.5. We know that \( s_\lambda, s_\mu, t_\lambda, t_\mu, \varepsilon_\lambda, \varepsilon_\mu, m_\lambda \) and \( m_\mu \) are group homomorphisms. So the conditions (i), (iv) and (vii) holds.

**ii** Let \((l,n) \in L \Lambda N \). Then
\[ v_{s_\lambda}((l,n)) = \nu(n) = s_\lambda((\lambda(l), \nu(n)) = s_\mu((\lambda \times \nu)(l,n)) \]
and
\[ v_{t_\lambda}(l,n) = \nu(\lambda'(l) + n) = \nu(\lambda'(l)) + \nu(n) = \mu(\lambda(l)) + \nu(n) = t_\mu((\lambda(l), \nu(n))) = t_\mu((\lambda \times \nu)(l,n)) \]

**iii** Let \((l,n) \in L \Lambda N \) and \((m,p) \in M \Lambda P \). Then
\[ s_\lambda((m,p) \cdot (l,n)) = p \cdot n = s_\lambda(m,p) \cdot s_\lambda(l,n) \]
and
\[ t_\lambda((m,p) \cdot (l,n)) = \lambda'(m \cdot (p \cdot l) + h(m,p \cdot n)) + p \cdot n = t_\lambda(m,p) \cdot t_\lambda(l,n). \]

**v** Let \( n \in N \). Then
\[ \alpha(l_\mu(n), \varepsilon_\lambda(n)) = \alpha_1(0,n) = (\lambda \times \nu)(0,n) = (\lambda(0), \nu(n)) = \varepsilon_\mu(v(n)) = \varepsilon_\mu(\alpha_\lambda(n)). \]

**vi** Let \( n \in N \) and \( p \in P \). Then
\[ \varepsilon_\lambda(p \cdot n) = (0,p \cdot n) = (0,p) \cdot (0,n) = \varepsilon_\mu(p \cdot \varepsilon_\lambda(n)). \]

**viii** Let \((l,n),(l',n') \in L \Lambda N \) such that \( n' = \lambda'(l) + n \). Then
\[ \alpha_\mu(m_\lambda((l',n'),(l,n))) = \alpha_\mu((l',n') \circ (l,n)) = (\lambda \times \nu)((l' + l,n)) = (\lambda(l' + l), \nu(n)) = (\lambda(l') + \lambda(l), \nu(n)) = m_\mu((\lambda(l), \nu(n))) = m_\mu((\lambda(l'), \nu(n))) = m_\mu((\lambda \times \nu)(l,n)). \]

**ix** Let \((m,p),(m',p') \in M \Lambda P \) such that \( n' = \lambda'(l) + n \) and \( p' = \mu(m) + p \). Firstly,
\[ h(m', m \cdot p \cdot n) = h(m - m + m' + m, p \cdot n) = m \cdot h((-m) \cdot m', p \cdot n). \]

Then
\[
((m', p') \circ (m, p)) \cdot ((l', n') \circ (l, n)) = (m' + m, p) \cdot (l' + l, n)
\]

and
\[
(m' + m, p) \cdot (l' + l, n) = ((m', p') \cdot (l', n')) \circ ((m, p) \cdot (l, n))
\]

Thus \( \psi(S) = C \) is an object in \( \text{Cat}(\text{XMod}) \). Now let
\[
f:S_1 = (L_1, M_1, N_1, P_1) \rightarrow S_2 = (L_2, M_2, N_2, P_2)
\]
be a morphism of crossed squares. Then \( \psi(f) = \{f_L \times f_N, f_M \times f_P, \{f_{L'}, f_{P'}\}: C \rightarrow C' \) is a morphism in \( \text{Cat}(\text{XMod}) \) where \( \psi(S_1) = C \) and \( \psi(S_2) = C' \).

Finally we show that composition of these functors are naturally isomorphic to the identity functors.

Example 3.12 Let \((A, B, \alpha)\) be a crossed module. Then the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow 1 & & \downarrow 1 \\
A & \xrightarrow{\alpha} & B
\end{array}
\]
has a structure of a crossed square where \( h(b, a) = b \cdot a - a \) for all \( a \in A \) and \( b \in B \).

Example 3.13 Let \((A, B, \alpha)\) be a crossed module. Then the diagram
\[
\begin{array}{ccc}
0 & \xrightarrow{0} & A \\
\downarrow \alpha & & \downarrow \alpha \\
0 & \xrightarrow{0} & B
\end{array}
\]
forms a crossed square where \( h(a, 0) = 0 \) for all \( a \in A \).

Example 3.14 Let \((A, B, \alpha)\) be a topological crossed module. Then we know that \((\pi A, A, s_A, t_A, e_A, m_A)\) and \((\pi B, B, s_B, t_B, e_B, m_B)\) are group-groupoids. Then
\[
\ker s_A \xrightarrow{\pi \alpha} \ker s_B
\]
has a crossed square structure where \( h([\beta], a) = [\beta \cdot a - a] \) for all \([\beta] \in \ker s_B \) and \( a \in A \).

Here the path \((\beta \cdot a - a): [0, 1] \rightarrow A\) is given by \((\beta \cdot a - a)(r) = \beta(r) \cdot a - a\) for all \( r \in [0, 1] \).

4. CONCLUSION

We proved that the category \( \text{Cat}(\text{XMod}) \) of internal categories within the category of crossed modules over groups and the category \( \text{X}^2\text{Mod} \) of crossed squares over groups are equivalent. Since crossed squares model all connected homotopy 3-types so are internal categories in within the category of crossed modules.

For further work, in a similar way of thinking one can obtain same results in a more generic algebraic category namely the category of groups with operations or in higher dimensional crossed modules [10]. Also in the light of the results given in [18], notions of normal subcrossed square and of quotient crossed square can be obtained.

REFERENCES


