Timelike Factorable Surfaces in Minkowski Space-Time

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ABSTRACT

In this study, we discuss timelike factorable surfaces in Minkowski 4 – space $\IE^4_1$. We calculate Gaussian and mean curvatures of these surfaces and classify timelike flat and minimal factorable surfaces in Minkowski space-time.

Keywords: factorable surface, timelike surface, Minkowski 4 – space

1. INTRODUCTION

The Minkowski space defined by Lorentzian inner product is the mathematical structure in which Einstein’s special relativity theory is the most appropriately represented. Since the inner product is not always positively defined, curves and surfaces vary in this space. Spacelike vectors, curves and surfaces show similarity to the Euclidean space structure.

In n – dimensional semi-Euclidean space, the Lorentzian inner product with $t$ – index is defined by

$$g(X, Y) = -\sum_{i=1}^{n-1} x_i y_i + \sum_{j=1}^{n} x_j y_j$$

where $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$ [1]. The semi-Euclidean space defined by this metric is denoted by $\IE^a_1$. Especially, for 4 – dimensional case with index $t = 1$, the semi-Euclidean space $\IE^4_1$ is called Minkowski space-time. In this case, the Lorentzian metric (1) is expressed in the form of

$$g(X, Y) = -x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$$ (2)

Any arbitrary vector $X = (x_1, ..., x_4)$ is called timelike, null or spacelike if the Lorentzian inner product $g$ is negative definite, zero or positive definite, respectively. Then, the length of the vector $X \in \IE^4_1$ is calculated by

$$\|X\| = \sqrt{g(X, X)}$$ (3)

where $X = (x_1, ..., x_4) \in \IE^4_1$.

Let $S$ be a surface in four-dimensional Minkowski space $\IE^4_1$. Then, the surface is called timelike if the induced metric $g$ on $S$ is a metric with index 1. Minkowski 4 – space can be written by the direct sum of the tangent space and the normal space of $S$ at each point $p$:

$$\IE^4_1 = T_p S \oplus T_p^\perp S$$ (4)

Represented by $\tilde{\nabla}$ and $\nabla$ is the Levi-Civita connections on $\IE^4_1$ and $S$, respectively. Let $X_1$ and $X_2$ indicate the tangent vector fields and let $\xi$ indicates the normal vector field on $S$. Then, Gauss and Weingarten formulas are given by the followings:

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1, X_2} + h(X_1, X_2),$$ (5)

$$\tilde{\nabla}_{\xi} \xi = -A_\xi X_1 + D_{\xi} X_1 \xi,$$

where $h, D$ and $A_\xi$ are the second fundamental tensor, the normal connection and the shape operator with regard to $\xi$, respectively [2].

Let $S$ be a timelike surface in $\IE^4_1$ given by the parameterization $F(u, v), (u, v) \in U (U \in \IE^2)$ and

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where the functions \( h_{jk}^k \), i, j, k = 1,2 are given by

\[
\begin{align*}
    h_{11}^1 &= -\frac{c_{11}^1}{E}, \\
    h_{12}^1 &= \frac{Ec_{12}^1 - Fe_{11}^1}{EW}, \\
    h_{22}^1 &= \frac{-Ec_{22}^1 - 2EFc_{12}^1 + F^2c_{11}^1}{EW^2}.
\end{align*}
\]

These coefficients are entries of shape operator matrices. Gaussian curvature of a timelike surface \( S : F(u,v) \) by using the second fundamental form coefficients is defined by

\[
K = \sum_{i,j} h_{ii}^1 h_{jj}^1 - \left( h_{12}^1 \right)^2,
\]
(see, [3]).

The mean curvature vector field can be calculated by

\[
H = \frac{1}{2} trh. \text{ Therefore, if } S \text{ is a timelike surface, then the mean curvature vector field is}
\]

\[
H = \frac{1}{2} \left( -h(X_1, X_1) + h(X_2, X_2) \right),
\]

(14)

where \( \{X_1, X_2 \} \) is a local orthonormal frame of the tangent bundle such that \( g(X_1, X_1) = -1, g(X_2, X_2) = 1 \) [2].

Any surface is said to be flat (minimal), if its Gaussian curvature (mean curvature vector) vanishes [4].

Factorable surfaces (also known homothetical surfaces) in \( \mathbb{R}^3 \) can be parameterized, locally, as \( F(u,v) = (u,v,f(u)g(v)) \), where \( f \) and \( g \) are smooth functions [5, 6]. Some authors have considered factorable surfaces in Euclidean space and in semi-Euclidean spaces [6, 7, 8, 9, 10, 11, 12]. In [5], Van de Woestyne proved that the only minimal factorable non-degenerate surfaces in \( \mathbb{L}^3 \) are planes and helicoids.

In [13, 14], the authors gave the surface parametrization as

\[
F(u,v) = (u,v,z(u,v),w(u,v))
\]
(15)

and called it Monge patch in \( \mathbb{L}^4 \). Furthermore, in [14], the authors characterized this surface and gave some examples. Also, some surfaces and curves in four dimensional spaces can be found in [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

In the present study, we consider timelike factorable surfaces in Minkowski \( 4 - \)space. We characterize
such surfaces in terms of their Gaussian curvature and mean curvatures and give the conditions for such surfaces to become flat and minimal.

2. TIMELIKE FACTORABLE SURFACES IN $\text{IE}^4_1$

Definition 1: Let $S \subset \text{IE}^4_1$ be a surface in 4–dimensional Minkowski space. If in (15), we take $z = f'_1(u)g_1(v), \ w = f'_2(u)g_2(v),$

where $f'_1, f'_2, g_1, g_2$ are differentiable functions in $\text{IE}^4_1,$ then we can define a surface parameterization (Monge patch) which is called factorable surface in Minkowski 4–space.

Therefore, the parameterization of the factorable surface can be written as

$$F(u, v) = (u, v, f'_1(u)g_1(v), f'_2(u)g_2(v))$$

(16)

Let $S$ be a timelike factorable surface given by the parameterization (16) in $\text{IE}^4_1.$ Then, the tangent space of the surface is spanned by the vector fields

$$F_u = \frac{\partial F(u, v)}{\partial u} = (1, 0, f'_1(u)g_1(v), f'_2(u)g_2(v))$$

$$F_v = \frac{\partial F(u, v)}{\partial v} = (0, 1, f'_1(u)g_1(v), f'_2(u)g_2(v))$$

(17)

The first fundamental form coefficients are obtained as

$$E = g(F_u, F_u) = -1 + \left( f'_1g_1 \right)^2 + \left( f'_2g_2 \right)^2.$$  

$$F = g(F_u, F_v) = f'_1f'_1g_1g_1 + f'_2f'_2g_2g_2,$$  

(18)

$$G = g(F_v, F_v) = 1 + \left( f'_1g_1 \right)^2 + \left( f'_2g_2 \right)^2.$$  

Here, we suppose $E<0,$ namely the surface $S$ is timelike, and so $EG - F^2 < 0.$

The second derivatives of $F(u, v)$ are

$$F_{uu} = \left( 0, 0, f''_1(u)g_1(v), f''_2(u)g_2(v) \right),$$

$$F_{uv} = \left( 0, 0, f'_1(u)f''_1(v), f'_2(u)f''_2(v) \right),$$

$$F_{vv} = \left( 0, 0, f'_1(u)f''_1(v), f'_2(u)f''_2(v) \right).$$

(19)

Normal space of the timelike surface $S$ is spanned by the orthonormal vector fields

$$\xi_1 = \frac{1}{\sqrt{A}} \left( f'_1g_1, -f'_2g_2, 1, 0 \right),$$

$$\xi_2 = \frac{1}{\sqrt{AD}} \left( Af'_2g_2 - Bf'_1g_1, \right.$$  

$$\left. B, 0, 0 \right),$$

(20)

where

$$A = 1 - \left( f'_1g_1 \right)^2 + \left( f'_2g_2 \right)^2,$$  

$$B = -f'_1f'_2g_2g_2 + f'_1f'_2g_1g_2,$$  

$$C = 1 - \left( f'_1g_1 \right)^2 + \left( f'_2g_2 \right)^2,$$  

$$D = AC - B^2.$$  

Since $S$ is timelike surface in $\text{IE}^4_1$ with respect to chosen orthonormal frame, $A$ and $D$ are positive definite. By the use of the equations (8), (19), and (20), we calculate the second fundamental form coefficients

$$c_{11}^1 = \frac{f''_1g_1}{\sqrt{A}}, \ c_{12}^1 = \frac{f''_1g_1}{\sqrt{A}},$$  

$$c_{11}^2 = \frac{Af'_2g_2 - Bf'_1g_1}{\sqrt{AD}},$$  

$$c_{12}^2 = \frac{Af'_2g_2 - Bf'_1g_1}{\sqrt{AD}}.$$  

(22)

With the help of (12) and (22), the shape operator matrices are

$$\left[ \begin{array}{cc} h_{11}^1 & h_{12}^1 \\ h_{12}^1 & h_{22}^1 \end{array} \right], \ \left[ \begin{array}{cc} h_{11}^2 & h_{12}^2 \\ h_{12}^2 & h_{22}^2 \end{array} \right].$$

where the functions $h_{ij}^k$ are given by

$$h_{11}^1 = -\frac{f''_1g_1}{E\sqrt{A}}, \ h_{12}^1 = \frac{f'g_1 - Ef''_1g_1F}{EW\sqrt{A}},$$  

$$h_{12}^2 = -\frac{f'_1g_1E^2 - Ef''_1g_1EF + f''_1g_1F^2}{EW^2\sqrt{A}},$$  

$$h_{11}^2 = -\frac{Af''_2g_2 - Bf''_1g_1}{E\sqrt{AD}}.$$  

(23)
surface: of the following parameterizations
obtain the desired result.

Proof. The parametrization (16) in
Theorem 1.

2.1. Timelike flat factorable surfaces

Theorem 1. Let $S$ be a timelike factorable surface with the parametrization (16) in $\mathbb{R}^4$. Then, its Gaussian curvature is given by

$$ K = \frac{1}{D W^2} \begin{pmatrix} f_1' f_1 f_1 g_1 - f_1'^2 g_1 & f_1 f_1 f_2 g_2 & f_1 f_2 f_1 g_2 & f_2'^2 f_1 f_2 g_2 \\ f_1 f_2 f_1 g_2 & f_1 f_2 f_2 g_2 - 2 f_1 f_2' g_2 & f_2 f_2 g_2 & f_2'^2 f_2 g_2 \\ + f_1 f_2 g_2 & f_2 f_2 g_2 & f_1 f_1 g_1 - f_1'^2 g_1 & f_1 f_2 f_1 g_2 \\ + f_1 f_2 g_2 & f_2 f_2 g_2 & f_1 f_1 g_1 - f_1'^2 g_1 & f_1 f_2 f_1 g_2 \end{pmatrix}.$$  \hspace{1cm} (24)

Proof. By the use of the equations (13) and (23), we obtain the desired result.

Theorem 2. Let $S$ be a timelike factorable surface with the parameterization (16) in $\mathbb{R}^4$. If $S$ is given by one of the following parameterizations, then it is a flat surface:

1. $F(u, v) = \left( u, v, a_1 g_1(v), a_2 g_2(v) \right)$,
2. $F(u, v) = \left( u, v, b_1 f_1(u), b_2 f_2(u) \right)$,
3. $F(u, v) = \left( u, v, a_1 g_1(v), a_2 f_2(u) \right)$,
4. $F(u, v) = \left( u, v, b_1 f_1(u), b_2 g_2(v) \right)$,
5. $F(u, v) = \left( u, v, a_1 b_1, \exp(a_2 u + b_2) \exp(a_3 v + b_3) \right)$,
6. $F(u, v) = \left( u, v, a_1 b_1, \frac{1}{(a_2 u + b_2)^{1/2}}(a_3 v + b_3)^{1/2} \right)$.

(7) $F(u, v) = \left( u, v, \exp(a_1 u + b_1) \exp(a_3 v + b_3), \exp(a_2 u + b_2) \exp(a_3 v + b_3) \right)$.

(8) $F(u, v) = \left( u, v, f_1(u) \cos v, f_1(u) \sin v \right)$,
the function $f_i(u)$ satisfies
$$ u = \pm \sqrt{\frac{a_i f_i^2(u) + 1}{f_i^2(u) + 1}} - df_i(u)$$
where $i, j = 1, 2, i \neq j$ and $a_k, b_k, k = 1, \ldots, 4$ are real constants.

Proof. Let $S$ be a timelike factorable surface given with the parameterization (16) in $\mathbb{R}^4$. If $f_1'(u) = 0$, $f_1'(u) = 0$ or $g_1'(v) = 0$, $g_2'(v) = 0$ or $f_1'(u) = 0$, $g_1'(v) = 0$, then we obtain the cases (1), (2), (3) and (4).

If $f_1'(u) = 0$, $g_1'(v) = 0$, then we have
$$ f_1'' f_2 f_2 g_2 = f_1'^2 f_2 g_2 = 0. $$ \hspace{1cm} (25)

Let $p(u) = \frac{df_2}{du}$ and $q(v) = \frac{dg_2}{dv}$. By the use of (25), we can write
$$ f_2(u) p(u) \frac{dp}{df_2} g_2(v) q(v) \frac{dq}{dg_2} = p(u) q(v). $$ \hspace{1cm} (26)

Then we have differential equation
$$ \frac{f_2(u)}{p(u)} \frac{dp}{df_2} = \frac{q(v)}{g_2(v)} \frac{dq}{dg_2} = \lambda, $$ \hspace{1cm} (27)
where $\lambda$ is constant.

(1) If $\lambda = 1$, from (27) we have
$$ f_2(u) = \exp(a_2 u + b_2), $$
$$ g_2(v) = \exp(a_3 v + b_3). $$ \hspace{1cm} (28)
which gives the case (5).

(2) If \( \lambda \neq 1 \), from (27) we have

\[
\begin{align*}
\int f_2(u) & = (a_2 u + b_2)^{1/\lambda}, \\
g_2(v) & = (a_3 v + b_3)^{1/\lambda},
\end{align*}
\]

which gives the case (6).

Further, we assume \( f_1 f_i g_i | g_i - f_i f_i g_i | | f_i f_i g_i = 0 \) holds for \( i = 1 \) and \( i = 2 \). Then we get

\[
\begin{align*}
f_1(u) & = \exp(a_1 u + b_1), \\
g_1(v) & = \exp(a_2 v + b_2), \\
f_2(u) & = \exp(a_3 u + b_3), \\
g_2(v) & = \exp(a_4 v + b_4),
\end{align*}
\]

Substituting these functions into \( B = 0 \) and \( f_1 f_2 g_2 + f_1 f_2 g_2 - 2 f_1 f_2 g_2 = 0 \), we have

\[
a_4 = \frac{a_4 a_1}{a_j}, \quad i, j = 1, 2 \quad (i \neq j)
\]

which vanish Gaussian curvature of the surface. Thus, we obtain the case (7).

Also, if \( f_i(u) = \hat{f}_i(u) \) and \( g_i(v) = \hat{g}_i(v) \), then we get

\[
\begin{align*}
- f_i(u)f_i(u)(f_i(u) + 1) + (f_i(u))^2((f_i(u))^2 - 1) = 0
\end{align*}
\]

which gives the case (8).

**Example 1.** The surface with the parameterization

\[
F(u, v) = \begin{pmatrix} u, v, \exp(2u + 3) \exp(3v + 4), \\
\exp(u + 1) \exp(\frac{2}{3}v + 2) \end{pmatrix}
\]

is timelike flat factorable surface in \( \mathbb{IE}_4^1 \) and one can plot its projection to 3-dimensional space with the help of the maple command

\[
\text{plot3d}([u + v, z, w], u = a..b, v = c..d).
\]

**2.2. Timelike minimal factorable surfaces**

**Theorem 2.** Let \( S \) be a timelike factorable surface with the parameterization (16) in \( \mathbb{IE}_4^1 \). Then its mean curvature vector is given by

\[
\bar{\mathbf{H}} = \frac{f_1 f_2 g_2 + f_1 g_1 g_1 - 2 f_1 f_2 g_2}{2\sqrt{A}}
\]

**Proof.** By the use of the equations (11), (14) and (23), we obtain the desired result.

**Theorem 3.** Let \( S \) be a timelike factorable surface with the parameterization (16) in \( \mathbb{IE}_4^1 \). Then \( S \) is a minimal surface if and only if

\[
f_1 g_1 + f_2 g_2 - 2 f_1 f_2 g_2 = 0, \quad i = 1, 2
\]

**Proof.** Let \( S \) be a factorable surface in \( \mathbb{IE}_4^1 \). Since we can write the mean curvature vector as \( \mathbf{H} = -\mathbf{H}_1 \mathbf{e}_1 - \mathbf{H}_2 \mathbf{e}_2 \), for a minimal surface \( \mathbf{H}_1 = 0, \mathbf{H}_2 = 0 \). With reference to the previous theorem, we get (34). The converse statement is trivial.

**Theorem 4.** Let \( S \) be a timelike factorable surface with the parameterization (16) in \( \mathbb{IE}_4^1 \). If \( S \) is given by one of the following parameterizations, then it is minimal:

(1) \( F(u, v) = \begin{pmatrix} u, v, (a_1 u + a_2) b_1, (a_3 u + a_4) b_2 \end{pmatrix} \),

(2) \( F(u, v) = \begin{pmatrix} u, v, a_1 (b_1 v + b_2), a_2 (b_3 v + b_4) \end{pmatrix} \),

(3) \( F(u, v) = \begin{pmatrix} u, v, (a_1 u + a_2) b_1, a_3 (b_3 v + b_4) \end{pmatrix} \),

(4) \( F(u, v) = \begin{pmatrix} u, v, a_1 b_1, \\
(u + a_2) - 1 - \exp(b_2 v + b_3) \end{pmatrix} \),

(5) \( F(u, v) = \begin{pmatrix} u, v, a_1 b_1, \tan(a_2 u + a_3) (v + b_2) \end{pmatrix} \).

Figure 1 Projection of timelike flat factorable surface given by (31)
\[ F(u,v) = \begin{cases} 
  u,v, \\
  \frac{-1 - a_2^2 + \exp(\pm 2a_1(a,u + a_i))}{2a_i \exp(\pm 2a_1(a,u + a_i))} \cos v, \\
  \frac{-1 - a_2^2 + \exp(\pm 2a_1(a,u + a_i))}{2a_i \exp(\pm 2a_1(a,u + a_i))} \sin v 
\end{cases} \]

(6) \[ F(u,v) = \begin{cases} 
  u,v, \\
  (u + a_2) - 1 - \exp(b_i v + b_2), \\
  (u + a_1) - 1 - \exp(b_i v + b_2) 
\end{cases} \]

(7) \[ F(u,v) = \begin{cases} 
  u,v, \\
  tan(a_i u + a_2) (v + b_1), \\
  tan(a_i u + a_2) (v + b_2) 
\end{cases} \]

(8) \[ F(u,v) = \begin{cases} 
  u,v, \tan(a_i u + a_2) (v + b_1), \\
  \tan(a_i u + a_2) (v + b_2) 
\end{cases} \]

Let \( S \) be a timelike factorable surface with the parameterization (16) in \( \mathbb{R}^4 \). By the use of (34) with (18), we get,

\[
\begin{align*}
  f_i'' g_i' &= (1 + f_i^2 g_i' + f_i g_i'')g_i' \\
  f_i'' g_i &= -1 + f_i g_i' + f_i g_i'' \\
  -2f_i g_i' f_i g_i' &= 0, \quad i = 1,2.
\end{align*}
\]

If \( g_1'(v) = 0 \), \( g_2'(v) = 0 \) or \( f_1'(u) = 0 \), \( f_2'(u) = 0 \), we obtain the cases (1) and (2), respectively.

If \( f_1'(u) = 0 \), \( g_1'(v) = 0 \), \( i, j = 1,2, i \neq j \), then

\[
\begin{align*}
  f_1 g_1^2 + f_2^2 g_2^2 &= 0, \\
  f_2 g_2^2 &= 0.
\end{align*}
\]

Since \( E < 0 \) and \( G > 0 \), then we get \( f_1''(u) = 0 \), \( g_1'(v) = 0 \) and \( g_2''(v) = 0 \), \( f_2'(u) = 0 \) which congruent the case (3).

If \( f_1'(u) = 0 \), \( g_1'(v) = 0 \), from the equality (35) for \( i = 2 \), we get

\[
\begin{align*}
  f_1'' g_1'' &= -g_2''(v) \\
  f_1'' g_1'' &= f_2''(u) - f_2''(u) \\
  g_2'' g_2'(v) &= g_2''(v) - g_2''(v)
\end{align*}
\]

If \( f_2'(u) = 0 \) or \( g_2''(v) = 0 \) in (38), we obtain the cases (4) and (5).

If \( f_2''(u)g_2''(v) \neq 0 \) in (38), differentiating (38) with respect to \( u \) and \( v \), we have

\[
\begin{align*}
  g_2''(v) - g_2''(v) &= c \\
  f_2''(u) &= f_2''(u)
\end{align*}
\]

Thus, we can write

\[
\begin{align*}
  f_2''(u) &= (1 + c)f_2''(u) = m, \\
  g_2''(v) &= (1 - c)g_2''(v) = n.
\end{align*}
\]

If \( c = 1 \), \( c = -1 \), and \( c \neq \pm 1 \), then from the solution of (39), we obtain the case (9).

If \( f_1'(u) = f_1'(u) \) and \( g_1'(v) = \cos v, g_2'(v) = \sin v \), then we get
By the solution of this differential equation we obtain the case (6).

If \( f_1(u) = f_2(u) \), \( g_1(v) = g_2(v) \) in (35), then for \( i = 1 \) or \( i = 2 \), we find

\[
\frac{f''(u)}{f_1(u)} \frac{1 + f_1^2(u)}{g_1(v)} - \frac{f'(u)}{g_1(v)} \left( 1 + f_1^2(u) \right) = 0. 
\]

By the solution of this differential equation we obtain the case (6).

If \( f_1(u) = f_2(u) \), \( g_1(v) = g_2(v) \) in (35), then for \( i = 1 \) or \( i = 2 \), we find

\[
\frac{f''(u)}{f_1(u)} \frac{1 + f_1^2(u)}{g_1(v)} + \left( \frac{f_1''(u)}{f_1(u)} - \frac{f_1^2(u)}{g_1(v)} \right) 2g_1'(u) = 0.
\]

If \( f_1''(u) = 0 \) or \( g_1''(v) = 0 \) in (40), we obtain the cases (7) and (8). Also, if \( f_1''(u)g_1''(v) \neq 0 \), we obtain the case (10), which completes the proof.

REFERENCES


