A Note On Convergence of Nonlinear General Type Two Dimensional Singular Integral Operators

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Abstract

The object of this study is to present both the pointwise convergence and the rate of convergence of the nonlinear integral operators given by

\[ V_{\zeta}(x, y; f) = \iint_{\Omega} K_{\zeta}(t, s, x, y; f(t, s)) ds dt, \quad (x, y) \in \Omega, \quad \zeta \in E \]

where \( \Omega = (a, b) \times (c, d) \) is arbitrary bounded region in \( \mathbb{R}^2 \) or \( \Omega = \mathbb{R}^2 \), moreover, \( E \) is a set of nonnegative numbers, \( \zeta_0 \) is an accumulation point of \( E \), and the function \( f \) is Lebesgue-integrable function on \( \Omega \).

Keywords: Lebesgue point, nonlinear singular integral, Lipschitz condition, pointwise convergence.

1. INTRODUCTION

In [1], Musielak introduced a new type convergence problem using the nonlinear integral operators in the following form

\[ T_{ab}f(t) = \int_{a}^{b} K_{ab}(s - t, f(s)) ds, \quad t \in (a, b) \quad (1) \]

and by assuming the whose kernel \( K_{ab} \) satisfies the generalized Lipschitz condition.

By using this idea, Bardaro et al. ([1], [3]) and in [4], Karsli studied the special cases of equation (1) in some Lebesgue spaces. Also, in [5], Swiderski and Wachnicki gave the theorems on pointwise approximation of the operators of equation (1) in the class of integrable and periodic functions. For further informations on nonlinear integral operators, we mention some of studies as [1]-[12]. Also, the several approximation properties of many new type integral operators have been studied and discussed by some authors, see [13]-[15].

In this note, first, we present the pointwise convergence, and in the sequel, we give the rate of convergence of general type nonlinear two dimensional singular integral operators of the following type:

\[ V_{\zeta}(x, y; f) = \iint_{\Omega} K_{\zeta}(t, s, x, y; f(t, s)) ds dt, \]

\( (x, y) \in \Omega, \zeta \in E \quad (2) \)

under various assumptions on \( f(t, s) \) and \( (K_{\zeta})_{\zeta \in E} \).

By \( L_{1}(\Omega) \), we denote the class of all functions \( f(t, s) \) Lebesgue-integrable over the rectangle
\( \Omega = (a, b) \times (c, d) \) or \( \Omega = \mathbb{R}^2 \) and \( E \) is a set of positive numbers and \( \zeta_0 \) is an accumulation point of \( E \).

First, we shall give the basic concepts which are used in this paper.

**Definition 1.1.** A point \((x_0, y_0) \in \Omega\) is a \( \mu \)-generalized Lebesgue point of function \( f \in L_1(\Omega) \) if

\[
\lim_{(h,k) \to (0,0)} \frac{1}{\mu(h,k)} \int_0^h \int_0^k |f(x_0 + t, y_0 + s) - f(x_0, y_0)| \, ds \, dt = 0
\]

where \( 0 < h \leq b - a \), \( 0 < k \leq d - c \) and \( \mu(h,k) = \int_0^h \int_0^k \rho(t,s) \, ds \, dt \)

is non-negative function provided \( \rho(t,s) \) is nonnegative Lebesgue integrable function defined on \([0, b - a] \times [0, d - c]\) (see [16]).

We have created the following definition by taking advantage of the article [11] and [17].

**Definition 1.2.** If the family of functions \( (K_\zeta)_{\zeta \in E} \) is bimonotonically increasing with respect to \( \zeta \) on both \((x, x + \delta) \times (y, y + \delta)\) and \((y - \delta, y) \times (x - \delta, x)\). Similarly, \( T_\zeta(t,s;x,y) \) is bimonotonically increasing with respect to \( t \) on both \([x, x + \delta] \times (y - \delta, y)\) and \((x - \delta, x) \times [y, y + \delta]\).

**2. POINTWISE CONVERGENCE**

Now we shall prove the existence of the integral operators in equation (2) by the Theorem 2.1.

**Theorem 2.1.** If \( f \in L_1(\Omega) \), then for every \( \zeta \in E \), \( V_\zeta \in L_1(\Omega) \) and \( \|V_\zeta\|_{L_1(\Omega)} \leq M \|f\|_{L_1(\Omega)} \)

**Proof.** We define a function

\[
h(t,s) = \begin{cases} f(t,s), & (t,s) \in \Omega \\ 0, & (t,s) \in \mathbb{R}^2 \setminus \Omega \end{cases}
\]

Using Fubini’s Theorem (see, e.g., [18]) and conditions (a), (b) and (f) of class \( A \) we get the following inequalities:

\[
\|V_\zeta(x,y;f)\|_{L_1(\Omega)} = \left( \int_{\Omega} K_\zeta(t,s,x,y;f(t,s)) \, ds \right) \, dy \, dx \\
\leq \int_{\Omega} \left( \int_{-\infty}^{\infty} |h(t,s)| T_\zeta(t,s,x,y) \, ds \right) \, dy \, dx \\
\leq \|f\|_{L_1(\Omega)} \|T_\zeta\|_{L_1(\mathbb{R}^2)}
\]

The proof is completed for this case.
Now we assume that $\Omega = \mathbb{R}^2$. Following similar steps, as in the first case, we have
\[
\|V_\zeta(x, y; f)\|_{L_1(\Omega)} \leq \|T_\zeta\|_{L_1(\mathbb{R}^2)} \|f\|_{L_1(\mathbb{R}^2)} \leq M \|f\|_{L_1(\mathbb{R}^2)}.
\]
The proof is completed.

We shall show to pointwise convergence of the operator (2) at the $\mu$-generalized Lebesgue point.

For $C > 0$, let $P_C$ denote the set
\[
\left\{(x, y, \zeta) \in \mathbb{R}^2 \times E : \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} \{T_\zeta(t, s; x_0, y_0) \times \rho(|t-x_0|, |s-y_0|)\} dsdt < C\right\}
\]

**Theorem 2.2.** Let $(x_0, y_0)$ be a $\mu$-generalized Lebesgue point of function $f \in L_1(\Omega)$ and functions $K_\zeta$ satisfies the assumptions listed in class $\mathcal{A}$, then for any $C > 0$ and $(x, y, \zeta) \in P_C$
\[
\lim_{\zeta \rightarrow \zeta_0} V_\zeta(x_0, y_0; f) = f(x_0, y_0).
\]

**Proof.** Suppose that $(x_0, y_0) \in \Omega$ is the $\mu$-generalized Lebesgue point of function $f \in L_1(\Omega)$ and
\[
0 < x_0 - x < \frac{\delta}{2} \text{ and } 0 < y_0 - y < \frac{\delta}{2} \text{ for all } \delta > 0
\]
satisfying
\[
x_0 + \delta < b, x_0 - \delta > a, \]
\[
y_0 + \delta < d, y_0 - \delta > c.
\]
For the remaining cases, the proof follows a similar line. From Definition 1.1, for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $h$ and $k$ satisfying $0 < h, k \leq \delta$ the inequality:
\[
\int_{x_0}^{x_0+h} \int_{y_0-k}^{y_0} |f(t, s) - f(x_0 - y_0)| dsdt < \varepsilon \mu(h, k)
\]
holds.

By conditions (b) and (c) of class $\mathcal{A}$, and from equation (3) using the extension $g(t, s)$ of $f(t, s)$, we get the following inequality:
\[
|V_\zeta(x_0, y_0; f) - f(x_0, y_0)| 
\leq \int_{\Omega} |f(t, s) - f(x_0 - y_0)| T_\zeta(t, s; x_0, y_0) dsdt
\]

\[+ |f(x_0, y_0)| \int_{\mathbb{R}^2 \setminus \Omega} T_\zeta(t, s; x_0, y_0) dsdt
\]

\[+ \int_{\mathbb{R}^2} K_\zeta(t, s; x_0, y_0; f(x_0, y_0)) dsdt - f(x_0, y_0)\]

\[= I_1 + I_2 + I_3.
\]
The necessity of (c) and (d) of class $\mathcal{A}$ provides the $I_2 \rightarrow 0$ and $I_3 \rightarrow 0$ as $\zeta \rightarrow \zeta_0$, respectively.

Splitting $I_1$ into two parts, we get the following:
\[
I_1 = \int_{\Omega \setminus N_\delta} |f(t, s) - f(x_0, y_0)| T_\zeta(t, s; x_0, y_0) dsdt
\]

\[+ \int_{N_\delta} |f(t, s) - f(x_0, y_0)| T_\zeta(t, s; x_0, y_0) dsdt
\]

\[= I_{11} + I_{12}
\]
where $N_\delta = (x_0 - \delta, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)$ stands for the family of all neighborhoods of $(x_0, y_0)$ in $\mathbb{R}^2$.

For the integral $I_{11}$ we may write the following
\[
I_{11} = \int_{\Omega \setminus N_\delta} |f(t, s) - f(x_0, y_0)| T_\zeta(t, s; x_0, y_0) dsdt
\]

\[
\leq \left( \sup_{(t, s) \in \mathbb{R}^2 \setminus N_\delta(x, y, \zeta)} \right) T_\zeta(t, s; x_0, y_0) \times (\|f\|_{L_1(\Omega)} + f(x_0, y_0) |b - a| |d - c|)
\]
by condition (d), $I_{11} \rightarrow 0$ as $\zeta$ tends to $\zeta_0$.

Now, we focus on the integral $I_{12}$. It is easy to see that $I_{12}$ can be written in the following form:
\[
I_{12} = \left\{ \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} |f(t, s) - f(x_0, y_0)| L_\zeta(t, s; x_0, y_0) dsdt \right\}
\]

\[= I_{121} + I_{122} + I_{123} + I_{124}.
\]
We shall prove $I_{12} \rightarrow 0$ as $\zeta \rightarrow \zeta_0$. It is enough to show that the integrals $I_{121}, I_{122}, I_{123}$ and $I_{124}$ tend to zero as $\zeta \rightarrow \zeta_0$ on $P_C$.

Let us define a new function as such:
Then, for all \( t \) and \( s \) satisfying \( t - x_0 \leq \delta \) and \( y_0 - s \leq \delta \) we have 
\[
|G(t, s)| \leq \varepsilon \mu(t - x_0, y_0 - s) 
\]
(4)

The following equality holds for the integral \( I_{121} \):
\[
|I_{121}| = \left| \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} |f(t, s)| \, ds \, dt \right|
\]
\[
= \left| \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} |f(t, s)| \, ds \, dt \right| 
+ \left| \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} (\mathcal{L}S) \right| 
\]
\[
= \left| \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} G(t, y_0 - \delta) d \xi \left[ \mathcal{T}_{\xi}(t, y_0 - \delta; x_0, y_0) \right] \right| 
\]
\[
+ \left| \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} G(x_0 + \delta, s) d \xi \left[ \mathcal{T}_{\xi}(x_0 + \delta, s; x, y) \right] \right| 
\]
\[
+ \left| \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} G(x_0 + \delta, y_0 - \delta) \left[ \mathcal{T}_{\xi}(x_0 + \delta, y_0 - \delta; x, y) \right] \right| 
\]
from equation (4), we have the following inequality:
\[
|I_{121}| \leq \varepsilon \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} \mu(t - x_0, y_0 - s) \left| \mathcal{T}_{\xi}(t, s; x_0, y_0) \right| \, ds \, dt 
\]
\[
+ \varepsilon \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} \mu(t - x_0, \delta) \left| \mathcal{T}_{\xi}(t, y_0 - \delta; x_0, y_0) \right| \, ds \, dt 
\]
\[
\leq \varepsilon \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} \mu(t - x_0, y_0 - s) \left| \mathcal{T}_{\xi}(t, s; x_0, y_0) \right| \, ds \, dt 
\]
\[
+ \varepsilon \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} \mu(t - x_0, \delta) \left| \mathcal{T}_{\xi}(t, y_0 - \delta; x_0, y_0) \right| \, ds \, dt 
\]
\[
\leq \varepsilon \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} \mu(t - x_0, y_0 - s) \left| \mathcal{T}_{\xi}(t, s; x_0, y_0) \right| \, ds \, dt 
\]
\[
+ \varepsilon \mu(\delta, y_0 - s) \left| \mathcal{T}_{\xi}(x_0 + \delta, s; x_0, y_0) \right| \, ds \, dt 
\]
\[
+ \varepsilon \mu(\delta, \delta) \left| \mathcal{T}_{\xi}(x_0 + \delta, y_0 - \delta; x_0, y_0) \right| 
\]
\[
= \varepsilon \mu(\delta, y_0 - s) \left| \mathcal{T}_{\xi}(x_0 + \delta, s; x_0, y_0) \right| 
\]
\[
+ \varepsilon \mu(\delta, \delta) \left| \mathcal{T}_{\xi}(x_0 + \delta, y_0 - \delta; x_0, y_0) \right| 
\]
\[
= \varepsilon \mu(\delta, y_0 - s) \left| \mathcal{T}_{\xi}(x_0 + \delta, s; x_0, y_0) \right| 
\]
\[
+ \varepsilon \mu(\delta, \delta) \left| \mathcal{T}_{\xi}(x_0 + \delta, y_0 - \delta; x_0, y_0) \right| 
\]
which in view of the definition of the set \( P_c \) tends to zero as \( \zeta \to \zeta_0 \).

Thus the proof of the theorem is completed.

**Theorem 2.3.** Let \( (x_0, y_0) \in \mathbb{R}^2 \) be a \( \mu \)-generalized Lebesgue point of function \( f \in \mathcal{L}_1(\mathbb{R}^2) \) and function \( \mathcal{K}_\xi \) satisfies the assumptions listed in class \( \mathcal{A} \), then for any \( C > 0 \) and \( (x, y, \zeta) \in P_c \)
\[
\lim_{\zeta \to \zeta_0} V_\zeta(x_0, y_0; f) = f(x_0, y_0). 
\]

**Proof.** The proof can be shown analogous to the proof of Theorem 2.2.

## 3. RATE OF CONVERGENCE

In this part, we establish the rate of pointwise convergence which we got in the Section 2.

**Theorem 3.1.** Suppose that the hypothesis of Theorem 2.2 is satisfied. Let
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**REFERENCES**


