Extending property on EC-Fully Submodules

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Abstract

There are several generalizations of CS-modules in literature. One of the generalization is based on fully invariant submodules. Recall that a module M is called FI-extending if every fully invariant submodule is essential in a direct summand. We call a module EFI-extending if every fully invariant submodule which contains essentially a cyclic submodule is essential in a direct summand. Initially we obtain basic properties in the general module setting. For example, a direct sum of EFI-extending modules is EFI-extending. Again, like the FI-extending property, the EFI-extending property is shown to carry over to matrix rings.

Keywords: fully invariant, ec-fully submodule, FI-extending, extending

1. INTRODUCTION

In recent years, the theory of extending modules and rings and their generalizations has come to play an important role in the theory of rings and modules. Recall that a module M is called an extending (or CS) module if every submodule of M is essential in a direct summand of M (see [4], [9] or [10]).

One of the extremely useful generalization of CS concept is FI-extending property (see [1] or [2]). Recall a module M is called FI-extending if every fully invariant submodule of M is essential in a direct summand. Following [3] and [5], by an ec-fully submodule N of a module M, we mean a fully invariant submodule N which contains essentially a cyclic submodule i.e., there exists an element x in N such that xR is essential in N.

In this paper, we are concerned with the study of modules M that every ec-fully submodule is essential in a direct summand of M. We call such a module as EFI-extending. Moreover, a ring R is called right EFI-extending ring if R_R is an EFI-extending module. Clearly the notion of an EFI-extending module generalizes that of a FI-extending module by requiring that only every ec-fully submodule is essential in a direct summand rather than every fully invariant submodule.

In Section 2, we provide basic properties of ec-fully submodules. After defining EFI-extending modules, in Section 3 we prove basic results and properties of EFI-extending modules. It is shown that any direct sum of EFI-extending modules is EFI-extending and that the EFI-extending property of a ring R carries over to the full matrix ring M_n(R), n ≥ 1.

Throughout this paper, all rings are associative with unity and R denotes such a ring. All modules are unital right R-modules.

Recall that a submodule X of M is called fully invariant if for every α ∈ End_R(M), α(X) ⊆ X. If M is an R-module and A ⊆ M, then we use A ⊆ M, A ≤_e M, A ≤ M, A ≤_e M, and E(M) to denote that A is a submodule, essential submodule, fully

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invariant submodule, $ec$-fully submodule, and the injective hull of $M$, respectively.

Moreover $M_n(R)$ denotes the full ring of $n$-by-$n$ matrices over $R$. For other terminology and notation, we refer to [2], [4], [7] and [10].

2. EC-FULLY SUBMODULES

Since $ec$-fully submodules are building bricks to the establishment of $EF1$-extending notion; first, we deal with this kind of submodules. To this end, we begin this section by recording some basic facts about them.

2.1. Lemma.

Let $M$ be a module.

(i) If $X \trianglelefteq_{ec} Y$ and $Y \trianglelefteq_{ec} M$ then $X \trianglelefteq_{ec} M$.

(ii) If $M = \bigoplus_{i \in \Lambda} X_i$ and $S \trianglelefteq_{ec} M$, then $S = \bigoplus_{i \in \Lambda} \pi_i(S) = \bigoplus_{i \in \Lambda} (S \cap X_i)$, where $\pi_i$ is the $i^{th}$-projection homomorphism of $M$.

Proof. The proof is routine.

The class of $ec$-fully submodules is properly contained in the class of fully invariant submodules. Next example provides a fully invariant submodule which is not $ec$-fully submodule. For details on this example, we refer to [8] or [10].

2.2. Example.

Let $\mathbb{R}$ be the real field and $S$ the polynomial ring $\mathbb{R}[x,y,z]$. Then the ring $R = S/SS_1$, where $s = x^2 + y^2 + z^2 - 1$, is a commutative Noetherian domain. The free $R$-module $M = R \oplus R \oplus R$ contains an indecomposable submodule $X_R$ of uniform dimension 2.

Now, let us build up the trivial extension of $R$ with $X_R$, i.e., let

$$T = \begin{bmatrix} R & X \\ 0 & R \end{bmatrix} = \left\{ \begin{bmatrix} r & x \\ 0 & r \end{bmatrix} : r \in R, x \in X \right\}.$$

Then $N = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \not\subseteq T_T$ but $N$ is not $ec$-fully submodule of $T_T$.

2.3. Lemma.

Let $M$ be a module which contains essentially a cyclic submodule. If $K$ is a fully invariant direct summand of $M$, then $K$ is an $ec$-fully submodule of $M$.

Proof. Suppose $Y = xR$ is an essential submodule of $M$, where $x \in M$. Let $\pi: M \rightarrow K$ be the canonical projection map. Then $xR \cap K = Y \cap K \leq \pi(Y) = \pi(x)R \leq K$. Since $xR$ is essential in $M$ then $xR \cap K$ is essential in $K$. It follows that $\pi(x)R$ is essential in $K$. Hence $K$ is an $ec$-fully submodule of $M$.
It is natural to think of which modules (even rings) have the property that every ec-fully submodule is a direct summand. Next result provides a class of rings which satisfy the aforementioned property. First, recall the following module condition:

\[ C_2 : \text{If } X \leq M \text{ is isomorphic to a direct summand of } M, \text{ then } X \text{ is a direct summand of } M \text{ (see } [4] \text{ or } [10]). \]

It is well-known that (von Neumann) regular rings satisfy the \( C_2 \) condition (see, for example [7]).

### 2.4. Proposition

Let \( R \) be a (von Neumann) regular ring. Then an ec-fully submodule of \( R \)-module \( R \) is a direct summand.

**Proof.** Let \( I \) be an ec-fully submodule of \( R \). Then there exists \( x \in I \) such that \( xR \) is essential in \( I \). By assumption, \( xR \) is a direct summand of \( R \). Thus \( R = xR \oplus L \) for some \( L \leq R \). Now \( xR \cap L \) is essential in \( I \cap L \) which yields that \( I \cap L = 0 \). Therefore \( R = xR \oplus L = I \oplus L \). It follows that \( I \cong xR \). Since \( R \) has \( C_2 \) condition, \( I \) is a direct summand of \( R \) as required.

### 3. EFI-EXTENDING MODULES

In this section, we define and obtain basic properties of EFI-extending modules. Let us start by mentioning the definition of this new class of modules.

#### 3.1. Definition

A module \( M \) is called **EFI-extending** if every ec-fully submodule of \( M \) is essential in a direct summand of \( M \).

Obviously \( FI \)-extending modules (and hence extending modules) are EFI-extending modules. Moreover, (von Neumann) regular rings enjoy with the EFI-extending property. On the other hand, the ring of integers is an EFI-extending ring which is not regular. One might expect that whether EFI-extending property implies \( FI \)-extending or not? However, the following examples show that the class of \( FI \)-extending modules are properly contained in the class of EFI-extending modules.

#### 3.2. Example

Let \( F \) be any field and let \( F_i = F, \ i \in \Lambda, \) where \( \Lambda \) is infinite. Define \( R = \bigoplus_{i \in \Lambda} F_i + F_1 \), which is an \( F \)-subalgebra of \( \Pi_{i \in \Lambda} F_i \), where \( 1 \) is the identity of \( \Pi_{i \in \Lambda} F_i \). It is known that \( R \) is a regular (and hence EFI-extending ring by Proposition 2.4) ring which is not \( FI \)-extending (see [2, Ex. 2.3.32]).

#### 3.3. Example [7, Ex. 7.54]

Let \( F \) be a field, and let \( A = F \times F \times \cdots \). So this ring is commutative. Now, let \( R \) be the subring of \( A \) consisting of sequences \((a_1, a_2, \ldots) \in A\) that are eventually constant. For any \((a_1, a_2, \ldots) \in R\), define \( x = (x_1, x_2, \ldots) \) by \( x_n = a_n^{-1} \) if \( a_n \neq 0 \), and \( x_n = 0 \) if \( a_n = 0 \). Then \( x \in R \) and \( a = axa \). Therefore, \( R \) is (von Neumann) regular. By Proposition 2.4, \( R \) is \( EFI \)-extending. Note that \( R \) is not a Baer ring. Hence \( R \) is not an \( FI \)-extending ring by [1, Theorem 4.7(iii)].

It is an open problem to determine if a direct summand of an \( FI \)-extending (or, also \( EFI \)-extending) module is always \( FI \)-extending (\( EFI \)-extending) (see [2]). The following result is in related with the \( EFI \)-extending version of the aforementioned problem.

#### 3.4. Proposition

Let \( M \) be a module and \( X \leq_{ec} M \). If \( M \) is \( EFI \)-extending, then \( X \) is \( EFI \)-extending.

**Proof.** Assume \( M \) is \( EFI \)-extending module. Let \( S \leq_{ec} X \). By Lemma 2.1 (i), \( S \leq_{ec} M \). Hence there exists a direct summand \( D \) of \( M \) such that \( S \leq_{ec} D \). Let \( \pi : M \to D \) be the canonical projection endomorphism. Then \( S = \pi(S) \leq \pi(X) \cap D = \pi(X) \). Hence \( S \leq_{ec} \pi(X) \) and \( \pi(X) \) is a direct summand of \( X \).

Next result deals with characterization of \( EFI \)-extending modules in terms of endomorphisms of injective hulls of the modules and complements of \( ec \)-fully submodules. To this end, the proof of the next theorem is based on the proof of the corresponding result for \( FI \)-extending modules (see [2, Proposition 2.3.2]).
3.5. Theorem

Let $M$ be a module. Then the following are equivalent:

(i) $M$ is EFI-extending

(ii) For $X \subseteq \text{ec}M$, there is $e^2 = e \in \text{End}(E(M))$ such that $X \leq_e eE(M)$ and $eM \leq M$.

(iii) Each $X \subseteq \text{ec}M$ has a complement which is a direct summand.

Proof. (i) $\Rightarrow$ (ii). Assume that $X \subseteq \text{ec}M$. Then there is $f^2 = f \in \text{End}(M)$ such that $X \leq_f fM$. Let $e : E(M) \to E(fM)$ be the canonical projection. Then we see that $X \leq_e eE(M)$ and $eM = fM \leq M$.

(ii) $\Rightarrow$ (iii). Let $X \subseteq \text{ec}M$. Then there exists $e^2 = e \in \text{End}(E(M))$ such that $X \leq_e eE(M)$ and $eM \leq M$. Now, let us put $c = (1-e)|_M$. Then $c^2 = c \in \text{End}(M)$. We show that $cM$ is a complement of $X$. For this, first note that $cM \cap X = 0$ as $cM = (1-e)M$. Say $K \leq M$ such that $cM = (1-e)M \leq K$ and $K \cap X = 0$. From $M = (1-e)M \oplus eM$, $K = (1-e)M \oplus (K \cap eM)$ by the modular law. As $K \cap X = 0$ and $X \subseteq_e eE(M)$, $K \cap eE(M) = 0$ and so $K \cap eM = 0$. Thus, we get that $K = (1-e)M$, then $K = cM$. Therefore $cM$ is a complement of $X$.

(iii) $\Rightarrow$ (i). Let $X \subseteq \text{ec}M$. There exists $g^2 = g \in \text{End}(M)$ so that $gM$ is a complement of $X$. As $X \subseteq \text{ec}M$, $gX \leq X \cap gM = 0$. Hence $X = (1-g)X$. To show that $M$ is EFI-extending, we claim that $X \leq_{g} (1-g)M$. For this, assume that $K \leq (1-g)M$ such that $X \cap K = 0$. Then note that $gM \cap K = 0$. Take $gm + k = n \in (gM \oplus K) \cap X$ with $m \in M$, $k \in K$, and $n \in X$. Then $(1-g)gm + (1-g)k = (1-g)n$, so $k = n \in X \cap K$ because $K \leq (1-g)M$ and $X = (1-g)X$. Now as $X \cap K = 0$, $k = n = 0$. Thus, $(gM \oplus K) \cap X = 0$. Since $gM$ is a complement of $X$, $gM \oplus K = gM$ and so $K = 0$. Therefore, $X \leq_{g} (1-g)M$. It follows that $M$ is EFI-extending.

It is well-known that a direct sum of EFI-extending modules is also EFI-extending module. Now, we intend to have the corresponding result for EFI-extending modules.

3.6. Theorem

Let $M = \bigoplus_{i \in A} N_i$. If each $N_i$ is an EFI-extending module, then $M$ is an EFI-extending module.

Proof. Let $S \subseteq \text{ec}M$. By Lemma 2.1(ii), $S = \bigoplus_{i \in A} (S \cap N_i)$, and $S \cap N_i \leq N_i$ for each $i \in A$. Assume $S$ contains essentially the cyclic submodule $xR$, where $x \in S$. Let $\pi : S \to S \cap N_i$ be the projection map. Then $xR \cap (S \cap N_i) \leq \pi(xR) = \pi(x)R \leq S \cap N_i$. Since $xR \leq_e S$ then $xR \cap (S \cap N_i) \leq_{e} S \cap N_i$. It follows that $\pi(x)R \leq_e S \cap N_i$. Hence $S \cap N_i \leq_{\text{ec}} N_i$ for each $i \in A$. As $N_i$ is EFI-extending, there is a direct summand $D_i$ of $N_i$ with $S \cap N_i \leq D_i$ for every $i \in A$. Thus $S = \bigoplus_{i \in A} (S \cap N_i) \leq_{e} \bigoplus_{i \in A} D_i$. Since $\bigoplus_{i \in A} D_i$ is a direct summand of $M$ we have that $M$ is an EFI-extending module.

3.7. Corollary

If $M$ is a direct sum of FI-extending (e.g., extending) modules, then $M$ is EFI-extending.

Proof. Immediate by Theorem 3.6.

Applying Theorem 3.6 to Abelian groups (i.e., $\mathbb{Z}$-modules) we obtain the following corollary.

3.8. Corollary

Let $M$ be a $\mathbb{Z}$-module. If $M$ satisfies any of the following conditions, then $M$ is an EFI-extending $\mathbb{Z}$-module.

(i) $M$ is finitely generated

(ii) $M$ is of bounded order (i.e., $nM = 0$ for some positive integer $n$)

(iii) $M$ is divisible.

Proof. (i) and (ii) $M$ is a direct sum of uniform submodules. Then the result follows from Theorem 3.6.

(iii) $M$ is extending and hence $FI$-extending. Thus $M$ is EFI-extending.

Observe that an easy modification yields that the Corollary 3.8 above remains true when the ring of integers replaced with a Dedekind domain.
problem of $\text{EFI}$-extending Abelian groups which as follows.

3.9. Theorem

Let $M$ be a direct sum of uniform $\mathbb{Z}$-modules. Then any direct summand of $M$ is an $\text{EFI}$-extending module.

Proof. Let $N$ be a direct summand of $M$. Then $N$ is also a direct sum of uniform modules by [9, Theorem 4.45] (see, also [10]). Now, Theorem 3.6 gives that $N$ is an $\text{EFI}$-extending module.

Our next objective is to carry over $\text{EFI}$-extending property to full matrix ring. First of all, we give an example of $\text{EFI}$-extending ring which shows that $\text{EFI}$-extending property is not left-right symmetric.

3.10. Example

Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$. Then the ring $R$ is right $\text{EFI}$-extending, but it is not left $\text{EFI}$-extending.

Proof. Note that $R$ is right $\text{FI}$-extending by [2, Example 2.3.14]. Hence $R$ is right $\text{EFI}$-extending ring. On the other hand, let $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} \leq_{\text{ec}} R$. It is easy to check that $I$ is not essential in a direct summand of $\mathfrak{m} R$. It follows that $R$ is not left $\text{EFI}$-extending ring.

3.11. Theorem

Let $R$ be a right $\text{EFI}$-extending ring. Then $M_n(R)$ is a right $\text{EFI}$-extending ring for all positive integer $n$.

Proof. Let $N \leq_{\text{ec}} M_n(R)$. Then it is easy to see that $N = M_n(I)$ for some $I \leq_{\text{ec}} R$. As $R$ is right $\text{EFI}$-extending, there exists $e^2 = e \in R$ such that $I_R \leq_{\text{e}} e R_R$. This yields that as a right ideal of $M_n(R)$, $N$ is essential in a direct $(eI)M_n(R)$ of $M_n(R)$, where $I$ is the identity matrix of $M_n(R)$. Thus $M_n(R)$ is right $\text{EFI}$-extending, as required.

REFERENCES


