Asymptotic Analysis of an Affine Transformation in the Supply of Missing Data

Hülya Kodal Sevindir*, Cüneyt Yazıcı1, Süleyman Çetinkaya1

ABSTRACT

Supply of missing data, also known as inpainting, is an important application of image processing. Wavelets are commonly used for inpainting algorithms. Shearlet transform which is an affine transformation is the improvement of the wavelet transform. An asymptotic analysis may help to evaluate the performance of an algorithm. In this article we compare the asymptotical analysis for wavelet and shearlet transforms in the case of inpainting where the missing data is shaped like a rectangle.

Keywords: Shearlet transform, wavelet transform, inpainting, asymptotic analysis

1. INTRODUCTION

Efficient representation of multidimensional data is an important issue which is an active research area [1-4]. Among these studies, Shearlet transform, introduced in 2006 by Guo et al., is a mathematical transform obtained as an extension of wavelets which is well-known as a good representation of one-dimensional data. [5,6] One of the most valuable properties of shearlets is that in order to control directional selectivity, it has the shearing parameter instead of the direction parameter in the curvelets. Due to this difference, shearlet transform can be represented by only one or a finite number of generator functions. That is why it presents optimal sparse representation for the multidimensional data. Besides, we can use shearlets for functions with finite support, and because of this transformation, we can obtain fast/superfast and effective algorithms. [10-14]

Inpainting problem is an inverse problem which is mainly concerned with finding some missing data in a signal or image. Missing data issue is a common problem in real life, and inpainting has many application areas: removing scratches or unwanted overlaid texts and graphics from old photos, or in general any image, etc. Some of the recent publications are as following: In [15], Häuser and Ma uses a shearlet based algorithm to recover missing data from seismic data. In [16], King et al. studied data separation and reconstruction by using clustered sparsity. In [17], King, Kutyniok, and Zhuang examined inpainting problem via clustered sparsity and showed an asymptotic analysis for the issue presented in [15]. In [18], King et al. considered the inpainting problem with missing data having

* Corresponding Author: hkodal@kocaeli.edu.tr
1 Department of Mathematics, Faculty of Arts and Sciences, University of Kocaeli, Turkey
2 Department of Mathematics Education, Faculty of Education, University of Kocaeli, Turkey

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different shapes except for horizontally positioned rectangle shape; but they did not show an asymptotic analysis. In this article we consider the inpainting problem which has missing data in a horizontally positioned rectangle shape and presents an asymptotic analysis for wavelet and shearlet transforms for this particular case. [19]

In Section 2 we give brief information on the theory of wavelets and shearlets. In Section 3 we introduce basic definitions and theorems which will be used for asymptotic analysis along with reconstruction model. In Section 4 we do the asymptotic analysis of the shearlet transform used for the particular inpainting problem described in Section 3.

2. PRELIMINARIES

2.1. Wavelet Transform

For 2-D wavelets let \( \gamma \in L^2(\mathbb{R}^2) \). Let the continuous affine systems of L2(\mathbb{R}^2) be defined as \( \gamma_{N,s} = T_s D_N^{-1} \gamma = |\det N| \gamma(N(-s)); (N,s) \in F \times \mathbb{R}^2 \). GL2(\mathbb{R}) is the group of invertible matrices and let F be a subset of it. Here \( D_N \) is the dilation operator on \( L^2(\mathbb{R}^2) \) determined by \( D_N \gamma(t) = |\det N|^{-1/2} \gamma(N^{-1}t), N \in GL2(\mathbb{R}), T_s \) is the translation operator on \( L^2(\mathbb{R}^2) \), defined by \( T_s \gamma(t) = \gamma(t-s), s \in \mathbb{R}^2 \). Any \( g \in L^2(\mathbb{R}^2) \) can be recovered from its coefficients \((g, \gamma_{N,s})_{N,s}\).

Therefore, one needs to discover requirements on \( \gamma \). We explain a group structure like \((N,s),(N',s') = (NN', s+Ns)\) to determine this. This group is said to be affine group on \( \mathbb{R}^2 \). It is denoted by \( A_2 \) [20].

Theorem 2.1. Let \( l_m \) be a left Haar measure of \( A_2 \) and \( l_1 \) be a left invariant Haar measure on \( F \subseteq GL2(\mathbb{R}) \). Moreover, suppose that \( \gamma \in L^2(\mathbb{R}^2) \) satisfies the admissibility condition \( \int_{\mathbb{R}^2} |\gamma(N^{T}\delta)|^2 |\det N| l_1(N) = 1 \). Then any function \( g \in L^2(\mathbb{R}^2) \) can be recovered via the reproducing formula \( g = \int_{\mathbb{R}^2} (g, \gamma_{N,s}) \gamma_{N,s} d\theta(N,s) \) explained weakly.

When the hypothesis of the above theorem are satisfied, \( \gamma \in L^2(\mathbb{R}^2) \) is called a continuous wavelet. Thus, \( L^2(\mathbb{R}^2) \) \( g \rightarrow W_f g(N,s) = (g, \gamma_{N,s}) \) is defined to be the Continuous Wavelet Transform.

2.2. Shearlet Transform

Shearlets has arisen in late times by various powerful applications. [21-24] shows some of the associated work. For produce waveforms with anisotropic support is required the scaling operator. Suppose that dilation operator like in wavelets. We will use the dilation operators \( D_b, b > 0 \), related to parabolic scaling matrices \( B_b = \begin{pmatrix} b & 0 \\ 0 & \sqrt{b} \end{pmatrix} \). The orientations of the waveforms can be changed by an orthogonal transformation. We select the shearing operator \( D_{c,e} \) \( c \in \mathbb{R} \), where the shearing matrix \( c \) is given by \( c = \begin{pmatrix} 1 & c \end{pmatrix} \) for orthogonal transformation. The shearing matrix uses variable \( c \) associated with the slopes. Lastly, \( T_e \) is used for the translation operator. Thus, continuous shearlet system \( SH(\gamma) \) can be defined by combining these 3 operators for \( \gamma \in L^2(\mathbb{R}^2) \):

\[
SH(\gamma) = \{ \gamma_{c,e,s} = T_e D_{c,e}^* D_b \gamma; b > 0, c \in \mathbb{R}, s \in \mathbb{R}^2 \}
\]

3. BASIC DEFINITIONS AND THEOREMS

In this section, we introduce some fundamental definitions and theorems which will be used later. Meyer wavelet function will be used for wavelet transformation. Auxiliary function \( v \in C^\infty(\mathbb{R}) \) which satisfies \( v() + v(1-) = \mathbb{I}_\mathbb{R}(\cdot) \) to form Meyer wavelet function is defined as

\[
v(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x \geq 1 \end{cases}
\]

Indicator function \( \mathbb{I}_\mathbb{R}(\cdot) \) is defined as \( \mathbb{I}_a(x) = 1, \quad x \in A \quad 0, \quad x \in A^c \). Meyer wavelet \( \psi(\omega) \) is defined as

\[
\psi(\omega) = \begin{cases} \frac{1}{2\pi} \sin \left( \frac{\omega}{2} \left( \frac{3|\omega|}{4\pi} - 1 \right) \right) e^{i\omega/2}, & 2\pi/3 < |\omega| < 4\pi/3 \\ \frac{1}{2\pi} \cos \left( \frac{\omega}{2} \left( \frac{3|\omega|}{4\pi} - 1 \right) \right) e^{i\omega/2}, & 4\pi/3 < |\omega| < 8\pi/3 \\ 0, & \text{otherwise} \end{cases}
\]

Fourier transformation of 1-D Meyer wavelet function is then

\[
W(\xi) = \begin{cases} e^{-\pi i \xi} \sin \left( \frac{\xi}{2} (3|\xi| - 1) \right), & 1/3 \leq |\xi| \leq 2/3 \\ e^{-\pi i \xi} \cos \left( \frac{\xi}{2} (3|\xi| - 1) \right), & 2/3 \leq |\xi| \leq 4/3 \\ 0, & \text{otherwise} \end{cases}
\]

1-D Meyer scaling function is

\[
\phi(\omega) = \begin{cases} \frac{1}{2\pi}, & |\omega| < 2\pi/3 \\ \frac{1}{2\pi} \cos \left( \frac{\omega}{2\pi} - 1 \right), & 2\pi/3 < |\omega| < 4\pi/3 \\ 0, & \text{otherwise} \end{cases}
\]

Fourier transformation of 1-D Meyer scaling function is
Fouier transformation for \( f \in L^1(\mathbb{R}^n) \) is \( \mathcal{F}f := \int_{\mathbb{R}^n} f(x)e^{-2\pi i \xi \cdot x} \, dx \). Here, \( \langle \cdot, \cdot \rangle \) stands for standard Euclidian inner product. Inverse Fourier transformation is defined as \( \mathcal{F}^{-1}f = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi)e^{2\pi i \xi \cdot x} \, d\xi \). When \( W^h \) stands for wavelet function to investigate horizontal mask case, function \( W^h \in C^\infty \cap L^2(\mathbb{R}^2) \) is defined as \( W^h(\xi) = W(\xi_1)\mathcal{F}(\xi_2) \). Orthonormal Meyer wavelet system is defined as \( \{\psi_d\}_d \); \( \lambda \in (0,\infty) \), \( d \in \mathbb{Z} \), \( k \in \mathbb{Z}^2 \) and function \( \psi_d(\xi) \) is defined as \( \psi_d(\xi) = 2^{-d/2}W^h(\xi/2^d)e^{-2\pi ik\xi/2^d} \), \( \lambda = (\ell, k, j) \). Parabolic scaling matrix \( A^h_\lambda \) and shear matrix \( S^h_\lambda \) are defined as \( A^h_\lambda = \begin{pmatrix} \alpha & 0 \\ \frac{1}{\sqrt{\lambda}} & 1 \end{pmatrix} \) and \( S^h_\lambda = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Shearlet function \( \delta^h \) is defined as \( \delta^h(\xi_1, \xi_2) = W(\xi_1)\mathcal{F}(\xi_2) \). Function \( V \in L^2(\mathbb{R}^2) \) satisfies \( V \in C^\infty(\mathbb{R}^2) \), supp \( V \subseteq [-1,1] \) and \( \frac{\lambda}{2 \pi} \sum_{n=\infty}^\infty |V(\xi + nk)^2| = 1 \). Let \( \mathcal{F} \) be an integer larger than or equal to \( d \). Here, \( \{x \} \) stands for an integer larger than or equal to \( x \).

### 3.1. Reconstruction Model

Modeling the reconstruction stage is highly important. To do so, let \( \mathcal{H} \) stand for Hilbert space, \( \mathcal{H}_\Lambda \) stand for the lost part and \( \mathcal{H}_\Lambda \) stand for the known part. Then we can write \( \mathcal{H} = \mathcal{H}_\Lambda \oplus \mathcal{H}_\Lambda \). For a given signal \( x^o \in \mathcal{H} \), the unknown part of \( x^o \) will be in the subspace \( \mathcal{H}_\Lambda \) and the known part of \( x^o \) will be in the subspace \( \mathcal{H}_\Lambda \). \( \mathcal{H}_\Lambda \) and \( \mathcal{H}_\Lambda \) show corresponding orthogonal projection transformations for these subspaces. In this case recovery problem is formulated as recovering \( x^o \) from the known \( \mathcal{P}_\Lambda x^o \). To do so, iterative thresholding will be used. During inpainting applications, recovered image sequences \( \{f_j\} \) will be obtained by \( \{f_j\} = (\mathcal{P}_\Lambda \mathcal{M}_h wL_j, f) \). Thresholding determination stage is done as follows: For thresholding value \( \beta_j \) at level \( j \), we consider the set \( J_j = \{i : \langle f_j, \Phi_i \rangle \geq \beta_j \} \) and apply iterative thresholding. In this case, recovered image at level \( j \) is obtained as \( L_j = \Phi_{J_j} \mathcal{P}_\Lambda wL_j \). Vectors \( \Phi = \{\phi_i\}_{i \in \mathcal{I}} \) in \( \mathcal{H} \) generates a Parseval frame for \( \mathcal{H} \) if for every \( x \in \mathcal{H} \), \( \mathcal{I}_i \|x, \phi_i\|_2^2 = \|x\|_2^2 \).

### Definition 3.1. [17] If \( \Phi \) is a Parseval frame and \( \Lambda \) is an index set of coefficients, then the concentration is defined on \( \mathcal{H}_\Lambda \) via \( \kappa(\mathcal{H}_\Lambda) = \sup_{f \in \mathcal{H}_\Lambda} \|\Delta f\|_\infty / \|\mathcal{P}_{\Lambda f}\|_1 \).

### Definition 3.2. [17] Let \( \Phi_1 = \{\phi_{i_1}\}_{i_1} \) and \( \Phi_2 = \{\phi_{i_2}\}_{i_2} \) be in \( \mathcal{H} \). Let \( \Lambda \subseteq \mathcal{I} \). Then the cluster coherence \( \mu_c(\Lambda, \Phi_1; \Phi_2) \) of \( \Phi_1 \) and \( \Phi_2 \) with respect to \( \Lambda \) can be defined by \( \mu_c(\Lambda, \Phi_1; \Phi_2) = \max_{i_1 \in \mathcal{I}} \sum_{i_2 \in \Lambda} |\langle \phi_{i_1}, \phi_{i_2} \rangle| \).

### Lemma 3.1. [17] The relation between the concentration \( \kappa(\mathcal{H}_\Lambda) \) and cluster coherence \( \mu_c \) can be obtained like that \( \kappa(\mathcal{H}_\Lambda) \leq \mu_c(\Lambda, \mathcal{P}_\Lambda \Phi ; \mathcal{P}_\Lambda \Phi) = \mu_c(\Lambda, \mathcal{P}_\Lambda \Phi ; \Phi) \).

### Lemma 3.2. [17] Let \( x^o \) and \( T \) be computed by the horizontal mask algorithm when \( \delta > 0 \). Consider that \( x^o \) is relatively sparse in \( \Phi \) with respect to \( T \). Then \( \|x^o - x^o\|_2 \leq c(\delta + \|\mathcal{P}_\Lambda \Phi x^o\|_1) \).

Let \( w : \mathcal{R} \rightarrow [0,1] \) be a smooth function having finite support in \( [-\rho, \rho] \). Let \( L \) show the real image and \( wL \) show the recovered image. We can use the following relation to see the effect of \( wL \) on the recovery model: \( \langle wL, f \rangle = \int_{\mathcal{R}} w(x) f(0, x) \, dx \).

Here, \( 2^p \) corresponds to the height of the horizontal rectangle (See Figure 4.1). Fourier transformation of \( wL \) is defined as \( \mathcal{F}(wL) = (wL, f) = \int_{\mathcal{R}} w(x) \int_{\mathcal{R}} f(\xi_1, \xi_2) d\xi_2 d\xi_1 \). Let \( f_1 \) be the filter corresponding to 2-D Meyer wavelet function and shearlet function at level \( j \). Fourier transformation of this filter is defined as \( f_j = \sum_{i \in \mathcal{H}} W^j (2^{-j} \xi) \). The filter of \( wL \) is denoted as \( wL_j \). Thus, we obtain \( wL_j = wL \ast f_j = \int_{\mathcal{R}} wL \mathcal{F}^{-1}(f_j(t)) \, dt \). This equation corresponds to cross-correlation. The lemma below let us to evaluate norm of \( wL_j \).

### Lemma 3.3. [17] For any \( c > 0 \), \( \|wL_j\|_2 \geq c^{1/2} \) is obtained as \( j \to \infty \).

### 4. ASYMPTOTIC ANALYSIS OF HORIZONTAL MASK APPLICATION

To get precise error analysis, we first apply a mask function to an image so that some parts of the data is missing. Let the function \( \mathcal{M}_h(x_1, x_2) = 1_{[|x_2|<\rho]} \) be horizontal mask function with height \( 2\rho \). The mask function \( \mathcal{M}_h \) is shown in Figure 4.1.
Lemma 4.1. Fourier transformation $\mathcal{M}_h$ of the horizontal mask function $\mathcal{M}_h$ can be written as $\mathcal{M}_h = 2hsinc(2h\xi)\delta(\xi)$.

Proof. The horizontal mask function $\mathcal{M}_h$ can be described by the Heaviside function as $\mathcal{M}_h(x_1, x_2) = H((x_1, x_2) + (0, h)) - H((x_1, x_2) - (0, h))$. From this equality, Fourier transformation $\mathcal{M}_h$ of the horizontal mask function $\mathcal{M}_h$ can be written as explicitly

$$\mathcal{M}_h = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-2\pi i x t}}{x} dx.$$

We can represent the optimal $\delta$-clustered sparsity by $\delta_j$ for filtered coefficients. Thresholding schemes will be analyzed by $\delta_j = \sum_{x \in T_j} |wL_j, \Psi_j|$ where the $T_j$ coefficients are obtained in the thresholding algorithm. The inpainting achieved on the filtered levels $j$ will be denoted by $L_j$. Here, we will denote the real filtered image by $wL_j; \text{that is, \ } wL_j * B_j$, where we will denote the original image by $L$. Thus, the basic theorems will show that

$$|\|L_j - wL_j\|_2^2 = 0, \ j \to \infty.$$

Here, the asymptotic behavior of the gap $h_j$ is important for these results.

Lemma 4.2. [17] For $j \to \infty$ and $h_j = o(2^{-j})$, thresholding values $b_j$ exist such that for $j \geq j_0$, $k: |k| \leq \rho 2^{(1+\alpha)j}, |k_j| \leq \rho 2^{2(\alpha)j} \in T_j$ holds.

Lemma 4.3. [17] For $j \to \infty$ we obtain $\delta_j = \sum_{x \in T_j} |wL_j, \Psi_j| = o(\|wL\|_2^2$).

Lemma 4.4. For $j \to \infty$ and $h_j = o(2^{-j})$ we obtain $\mathcal{M}_h | (ML_j, \Psi_j) = o(2^{j/2})$.

Proof. First let us evaluate the term $\langle \mathcal{M}_h, wL_j, \Psi_j \rangle$. Let $G_j$ be the inverse Fourier transformation of $F_j$. From the cross-correlation theorem and the property $G_j(x) = G_j(-x)$, we have

$$\langle \mathcal{M}_h, wL_j, \Psi_j \rangle = \langle \mathcal{M}_h, wL_j, \Psi_j \rangle.$$
\[
\begin{align*}
&= \left(2h_j \text{sinc}(2h_j \cdot) e^{-2\pi n_k z_2 / z_1}\right) Y(-x) \\
&= \left(\text{FW}^h(\xi_2, \cdot / 2^j)\right) Y(-x)
\end{align*}
\]

From here, if \( h_j < \rho \), we obtain

\[
\int_{t_{k_2} - h_j}^{t_{k_2} + h_j} \left| (\text{FW}^h((\xi_2, \cdot / 2^j))) Y(x) \right| dx = \int_{t_{k_2} - 2h_j}^{t_{k_2} + 2h_j} \left| h_{k_2} (\text{FW}^h((\xi_2, \cdot / 2^j))) Y(x) \right| dx.
\]

Thus by considering \((\text{FW}^h)((\cdot, \cdot)) Y(x) \leq c(\|x\|)^{-N_2}\), we obtain

\[
\|E\|_\infty \leq c \min\{k_2 - 2h_j, k_2 + 2h_j\}^{-N_2}.
\]

Combining all these considerations, we obtain

\[
\begin{align*}
&\|M_{h_j} wL_j, \Psi_k\| \leq c\{k_1\}^{-N_2} \min\{k_2 - 2h_j, k_2 + 2h_j\}^{-N_2}.
\end{align*}
\]

By using Lemma 4.2, as a result we obtain

\[
\begin{align*}
&\|M_{h_j} wL_j, \Psi_k\| \leq c\sum_{k_2 \in \mathbb{Z}} \min\{k_2 - 2h_j, k_2 + 2h_j\}^{-N_2}.
\end{align*}
\]

Expected convergence for normalized error \( \xi_2 \) of reconstructed filter \( L_j \) can be obtained using iterative method via wavelet transformation by the following theorem.

**Theorem 4.1.** Consider 2-D Meyer orthonormal system \( \Phi \) with the filter \( L_j \) for \( h_j = o(2^{-j}) \). Then \( \|L_j - wL_j\|_2 \to 0 \), \( j \to \infty \) holds.

**Proof.** By letting \( x^* = L_j \) and \( x^0 = wL_j \) in the Lemma 3.2, we obtain

\[
\begin{align*}
&\|L_j - wL_j\|_2 = \|\Theta \Phi_{j*, \Phi^* \Phi^* P_{R_k} \Phi \Psi_k - \Theta \Phi_{j*, \Phi^* P_{R_k} \Phi \Psi_k}\|_2 \\
&\leq \sum_{k_2 \in \mathbb{Z}} \left(\|wL_j \Phi_k\|_2 \right) \geq \sum_{k_2 \in \mathbb{Z}} \left(\|M_{h_j} wL_j, \Psi_k\|\right) \leq \frac{c}{N_2 \|w\|_2} \|L_j - wL_j\|_2.
\end{align*}
\]

From Lemma 3.3, we obtain

\[
\begin{align*}
&\frac{\|L_j - wL_j\|_2}{\|wL_j\|_2} \leq \frac{c}{N_2 \|w\|_2} \|L_j - wL_j\|_2 + c_2 2^{-j/2} \to 0, \; j \to \infty
\end{align*}
\]

Similar to the wavelet transformation case above, expected convergence for normalized error \( \xi_2 \) of reconstructed filter \( L_j \) can be obtained by the following steps: We consider the set \({\mathcal{T}_j = \{n = (n, l, k, k): |wL_j, \sigma_k| \geq \beta_j \}}\) of coefficients of thresholding values for \( \beta_j > 0 \).

**Lemma 4.5.** [17] For all \( j \geq j_0 \) and for some values \( j_0, v_1, v_2 < 1/4 \), thresholding coefficients \( \beta_j \) exist as follows: \({\{n; l, k, k\}: |k_1| \leq 2(1 + n_1), |k_2| \leq 2^{2v_2}, \; \ell = 0; v = \ell \} \subseteq \mathcal{T}_j \). For \( h_j = o(2^{-j/2}) \) when \( j \to \infty \).

**Lemma 4.6.** [17] When \( j \to \infty \) for \( h_j = o(2^{-j/2}) \), we obtain \( \sum_{k_2 \in \mathbb{Z}} \{M_{\mathcal{H}_j} wL_j, \sigma_k\} = o(2^{j/4}) \).

**Proof.** First, evaluation of \( \{M_{\mathcal{H}_j} wL_j, \sigma_k\} \) is needed. Similar to the proof of Lemma 4.4, by using definitions of \( M_{\mathcal{H}_j}, wL_j, \sigma_k \) and cross-correlation theorem, we obtain

\[
\begin{align*}
\{M_{\mathcal{H}_j} wL_j, \sigma_k\} &= 2^{j/4} \int \int \hat{w}(\xi_2) 2h_j \int \text{sinc}(2h_j \tau_2) F(\xi_2, 2\tau_2) \int \frac{\xi_2}{\tau_2} W(\xi_1, \xi_2) V(\xi_2, \tau_2, \xi_2) W(\xi_1, \tau_2, \xi_2) \int \frac{\xi_2}{\tau_2} d\tau_2 d\xi_2 d\xi_1 \int \frac{\xi_2}{\tau_2} d\tau_2 d\xi_2
\end{align*}
\]

The function \( \hat{G} \) is defined as

\[
\hat{G}(\xi_1) = \hat{w}(\xi_2) 2h_j \int \text{sinc}(2h_j \tau_2) F(\xi_2, 2\tau_2) \int \frac{\xi_2}{\tau_2} W(\xi_1, \xi_2) V(\xi_2, \tau_2, \xi_2) W(\xi_1, \tau_2, \xi_2) \int \frac{\xi_2}{\tau_2} d\tau_2 d\xi_2 d\xi_1 \int \frac{\xi_2}{\tau_2} d\tau_2 d\xi_2
\]

This function has finite support on the set \[1/2, 2\]. From here, we obtain \( \{M_{\mathcal{H}_j} wL_j, \sigma_k\} \leq c_{N_2} 2^{-2/4} \|L_j - wL_j\|_2 \).

Let us investigate the norm \( \|L_j - wL_j\|_2 \):
Thus finally when $v_2 < 1/4$ from Lemma 4.5, we obtain

$$
\frac{1}{c} \sum_{q \in \mathbb{F}_k^\prime} \| (\mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j) \| \leq 2^{j/4} \sum_{q \in \mathbb{F}_k^\prime} \| (k_{2j} \omega) \| \min \left\{ \| k_{2 - 1/2j} \|, k_{2 + 1/2j} \right\} \leq 2^{j/14 + v_2} \tag{28}
$$

As in the wavelet transformation case above, expected convergence for normalized error $\varepsilon_2$ of reconstructed filter $L_j$ can be obtained using iterative method via shearlet transformation by the following theorem.

**Theorem 4.2.** Consider filter $L_j$ for $h_j = o(2^{-1/2})$ with 2-D shearlet system $\Phi$. Then

$$
\| L_j - w \mathcal{L}_j \| = \| \Phi \left( \mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j \right) - \Phi \left( \mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j \right) \| \leq \| \sum_{q \in \mathbb{F}_k^\prime} \| (w \mathcal{L}_j, \sigma_j) \| \| (\mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j) \| \| \leq \| \sum_{q \in \mathbb{F}_k^\prime} \| (w \mathcal{L}_j, \sigma_j) \| \| (\mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j) \| \| \leq 2^{j/2} \tag{29}
$$

**Proof.** Similarly by letting $x^* = L_j$ and $x^0 = w \mathcal{L}_j$ in the Lemma 3.2, we obtain

$$
\| L_j - w \mathcal{L}_j \| = \| \Phi \left( \mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j \right) - \Phi \left( \mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j \right) \| \leq \| \sum_{q \in \mathbb{F}_k^\prime} \| (w \mathcal{L}_j, \sigma_j) \| \| (\mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j) \| \| \leq 2^{j/2} \tag{30}
$$

From Lemma 3.3, we obtain

$$
\| L_j - w \mathcal{L}_j \| \leq \| \Phi \| \| (w \mathcal{L}_j, \sigma_j) \| \| (\mathcal{M}_{h_j} w \mathcal{L}_j, \sigma_j) \| \| \leq 2^{j/2} \tag{31}
$$

Thus, we prove that the image can be reconstructed well asymptotically, when the height of the horizontal mask decays faster than $2^{-1/2}$.

**5. CONCLUSION**

In this paper, we show the asymptotic analysis of wavelet and shearlet transforms used for the inpainting where the missing data have a horizontal rectangle shape. As a conclusion, we found out that the shearlet transformation is more effective for the problem discussed than the wavelet transform. If the height of the horizontal mask decays faster than $2^{-1/2}$, we proved that the image can be reconstructed asymptotically.

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Asymptotic analysis of shearlet transform for inpainting…


