Continuous dependence of a coupled system of Wave-Plate Type

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ABSTRACT

In this study, we prove continuous dependence of solutions on coefficients of a coupled system of wave-plate type.

Keywords: Wave-plate type, continuous dependence.

Wave-Plate Tipi denklem sisteminin sürekli bağımlılığı

ÖZ

Bu çalışmada, wave-plate tipi denklem sisteminin çözümlerinin katsayılara sürekli bağımlılığı ispatlanmıştır.

Anahtar Kelimeler: Wave-plate tipi, sürekli bağımlılık.

1. INTRODUCTION

In this paper, we consider the following coupled system of wave-plate type:

\[ au_{tt} - \Delta u - \mu \Delta u_r + a \Delta v = 0, \quad x \in \Omega, \quad t > 0 \quad (1) \]

\[ \beta v_{rr} + \gamma \Delta^2 v + a \Delta u - h \Delta v_r = 0, \quad x \in \Omega, \quad t > 0 \quad (2) \]

\[ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \quad x \in \Omega, \quad (3) \]

\[ (u_r(x,0), v_r(x,0)) = (u_r(x), v_r(x)), \quad x \in \Omega, \quad (4) \]

\[ u = v = \frac{\partial v}{\partial v} = 0, \quad x \in \partial \Omega, \quad t > 0. \quad (5) \]

Here \( \Omega \) is a open set of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \); \( \alpha, \beta, \gamma, \mu, a \) and \( h \) are positive constants.

Continuous dependence of solutions of problems in partial differential equations on coefficients in the equations is a type of structural stability, which reflects the effect of small changes in coefficient of equations on the solutions. This type has been extensively studied in recent years for a variety of problems. Many results of this type can be found in the literature (see, 1-14, 16, 17, 20-22, 24). Most

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of the paper in the literature study structural stability for various systems in a finite region. For a review of such works, one can refer to [4, 18-20] and papers cited therein. Also, many papers in the literature have studied the Brinkman, Darcy, Forchheimer and Brinkman Forchheimer equations, see [2, 3, 8-16].

In [15], Santos and Munoz Rivera studied the analytic property and the exponential stability of the $C_0$-semigroup associated with the following coupled system of wave-plate type with thermal effect:

\begin{align}
\rho \ddot{u}_n - \Delta u - \mu \dot{u}_n + \alpha \Delta v &= 0, \\
\rho_2 \ddot{v}_n + \gamma \Delta^2 v + a \Delta u + m \Delta \theta &= 0, \\
\tau \dot{\theta} + k \Delta \theta - m \Delta v &= 0,
\end{align}

where the functions $u$ and $v$ represent the vertical deflections of the membrane and the plate, respectively, $\theta$ is the difference between the two temperatures and finally $\rho_1, \rho_2, \mu, \gamma, k, m$ and $\tau$ are positive constants. The above model can be used to describe the evolution of a system consisting of an elastic membrane and an elastic plate, subject to a thermal effect and attracting each other by an elastic force with coefficient $\alpha > 0$.

In 2014, Tang, Liu and Liao [23] studied the spatial behavior of the following coupled of the wave-plate type:

\begin{align}
\rho \ddot{u}_n - \Delta u - \mu \dot{u}_n + a \Delta v &= 0, \\
\rho_2 \ddot{v}_n + \gamma \Delta^2 v + a \Delta u - h \Delta v &= 0.
\end{align}

The authors got the alternative results of Phragmen-Lindelof type in terms of an area measure of the amplitude in question based on a first-order differential inequality. They also got the spatial decay estimates based on a second-order differential inequality.

Throughout in paper, $\| \|$ and $( , )$ denote the norm and inner product $L^2(\Omega)$.

2. A PRIORI ESTIMATES

**Theorem 1.** Let $u$ and $v$ be the solutions of the problem (1)-(5). Then the following estimate holds:

\begin{align}
\| u_n \|^2 &\leq D_1(t), \| v_n \|^2 \leq D_2(t), \\
\| \nabla u_n \|^2 &\leq D_3(t), \| \Delta v_n \|^2 \leq D_4(t),
\end{align}

where $D_1(t) = \frac{2}{\alpha} D_0(t)$, $D_2(t) = \frac{2}{\beta} D_0(t)$, $D_3(t) = 2 D_0(t)$, $D_4(t) = \frac{2}{\gamma} D_0(t)$, and $D_0(t)$ is a function depending on the initial data and the parameters of (1)-(2).

**Proof.** Firstly, we differentiate (1) and (2) with respect to $t$:

\begin{align}
\alpha u_{nn} - \Delta u_t - \mu \Delta u_n + a \Delta v_t &= 0, \\
\beta v_{nn} + \gamma \Delta^2 v + a \Delta u_t - h \Delta v_n &= 0.
\end{align}

Multiplying (12) and (13) by $u_n$ and $v_n$ in $L^2(\Omega)$, respectively we get

\begin{align}
\frac{d}{dt} E_1(t) + \mu \| \nabla u_n \|^2 + h \| \nabla v_n \|^2 = a (\nabla u_t, \nabla v_n) + a (\nabla v_t, \nabla u_n),
\end{align}

where

\begin{align}
E_1(t) = \frac{1}{2} \| u_n \|^2 + \frac{\beta}{2} \| v_n \|^2 + \frac{1}{2} \| \nabla u_t \|^2 + \frac{\gamma}{2} \| \Delta v_n \|^2.
\end{align}

Using the Cauchy's inequality with $\varepsilon$ and the Sobolev inequality two terms on the right hand side of (14) we obtain

\begin{align}
a (\nabla u_t, \nabla v_n) \leq \varepsilon_1 \| \nabla v_n \|^2 + \frac{a^2}{4 \varepsilon_1} \| \nabla u_t \|^2
\end{align}

and

\begin{align}
a (\nabla v_t, \nabla u_n) \leq \varepsilon_2 \| \nabla u_n \|^2 + \frac{a^2}{4 \varepsilon_2} \| \nabla v_t \|^2
\leq \varepsilon_2 \| \nabla u_n \|^2 + \frac{a^2}{4 \varepsilon_2} d_1 \| \Delta v_n \|^2,
\end{align}

where $d_1$ is the positive constant in the Sobolev inequality. From (15) and (16) with $\varepsilon_1$ and $\varepsilon_2$ are selected sufficiently small we obtain

\begin{align}
\frac{d}{dt} E_1(t) \leq M_1 E_1(t),
\end{align}

where $M_1$ is a positive constant depending on the parameters of (1) and (2). So
\[ \|u_n\| \leq \frac{2}{\alpha} D_0(t), \quad \|v_n\| \leq \frac{2}{\beta} D_0(t), \]
\[ \|\nabla u_n\| \leq 2D_0(t), \quad \|\nabla v_n\| \leq \frac{2}{\gamma} D_0(t), \]
where \( D_0(t) = E_i(0)e^{\mu t} \). Therefore (11) is satisfied.

### 3. CONTINUOUS DEPENDENCE ON PARAMETERS

In this section, we prove that the solution of the problem (1)-(5) depends continuously on \( \mu \) and \( h \).

Now assume that \((u_1, v_1)\) is the solution of the problem
\[
\begin{align*}
\alpha(u_1) - \Delta u_1 - \mu_1 \Delta u_1 + a \Delta v_1 &= 0 & \text{in } \Omega, \quad t > 0, \\
\beta(v_1) + \gamma \Delta^2 v_1 + a \Delta u_1 - h \Delta v_1 &= 0 & \text{in } \Omega, \quad t > 0,
\end{align*}
\]
\[(u_1(x, 0), v_1(x, 0)) = (u_0(x), v_0(x)) & \quad x \in \Omega,
((u_1), (x, 0), (v_1), (x, 0)) = (u_1(x), v_1(x)) & \quad x \in \Omega,
\]
\[u_1 = v_1 = \frac{\partial v_1}{\partial v} = 0, & \quad x \in \partial \Omega, \quad t > 0
\]
and \((u_2, v_2)\) is the solution of the following problem
\[
\begin{align*}
\alpha(u_2) - \Delta u_2 - \mu_1 \Delta u_2 + a \Delta v_2 &= 0 & \text{in } \Omega, \quad t > 0, \\
\beta(v_2) + \gamma \Delta^2 v_2 + a \Delta u_2 - h \Delta v_2 &= 0 & \text{in } \Omega, \quad t > 0,
\end{align*}
\]
\[(u_2(x, 0), v_2(x, 0)) = (u_0(x), v_0(x)) & \quad x \in \Omega,
((u_2), (x, 0), (v_2), (x, 0)) = (u_2(x), v_2(x)) & \quad x \in \Omega,
\]
\[u_2 = v_2 = \frac{\partial v_2}{\partial v} = 0, & \quad x \in \partial \Omega
\]
Let \( u = u_1 - u_2 \), \( v = v_1 - v_2 \) and \( \mu = \mu_1 - \mu_2 \). Then \((u, v)\) satisfies the problem
\[
\begin{align*}
\alpha(u) - \Delta u - \mu \Delta u_1 - \mu \Delta (u_2) &= 0 & \text{in } \Omega, \quad t > 0, \\
\beta(v) + \gamma \Delta^2 v + a \Delta u - h \Delta v &= 0 & \text{in } \Omega, \quad t > 0,
\end{align*}
\]
\[(u(x, 0), v(x, 0)) = (0, 0) & \quad x \in \Omega, \quad t > 0,
(u(x, 0), v(x, 0)) = (0, 0) & \quad x \in \Omega
\]
u = v = \frac{\partial v}{\partial v} = 0, \quad x \in \partial \Omega, \quad t > 0. \quad (22)

Firstly the following theorem establishes continuous dependence of the solution of (1)-(5) on the coefficient \( \mu \).

**Theorem 2.** Let \( u \) and \( v \) be the solutions of the problem (18)-(22). Then the following estimate holds:
\[
\|u\|^2 + \|v\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2 \leq (\mu_1 - \mu_2)^2 A_1(t), \quad \forall t > 0
\]
(23)

**Proof.** Multiplying (18) and (19) by \( u \) and \( v \) in \( L^2(\Omega) \), respectively and adding the obtained relations, we get
\[
\frac{d}{dt} E_2(t) + \mu \|\nabla u_1\|^2 + h \|\nabla v_1\|^2 + \mu (\nabla (u_2), \nabla u_1) - a(\nabla u_1, \nabla v) - a(\nabla v_1, \nabla u) = 0,
\]
where
\[
E_2(t) = \frac{\alpha}{2} \|u\|^2 + \frac{\beta}{2} \|v\|^2 + \frac{\gamma}{2} \|\Delta v\|^2 + \frac{1}{2} \|\nabla u\|^2.
\]
Using the Cauchy’s inequality with \( \varepsilon \) for sufficiently small \( \varepsilon_1 > 0, \varepsilon_2 > 0 \) and \( \varepsilon_3 > 0 \), we can write the following inequality:
\[
\frac{d}{dt} E_2(t) + (\mu_1 - \varepsilon_1 - \varepsilon_2) \|\nabla u_1\|^2 + (h - \varepsilon_3) \|\nabla v_1\|^2 \leq \frac{\mu_1^2}{2\varepsilon_1} \|\nabla (u_2)\|^2 + \frac{a^2}{2\varepsilon_2} \|\nabla v\|^2 + \frac{a^2}{4\varepsilon_3} \|\nabla u\|^2.
\]
(25)

Then there exist \( \mu_1 \geq \varepsilon_1 + \varepsilon_2 \) and \( h \geq \varepsilon_3 \) such that
\[
\frac{d}{dt} E_2(t) \leq \frac{\mu_1^2}{2\varepsilon_1} \|\nabla (u_2)\|^2 + \frac{a^2 d_2}{2\varepsilon_2} \|\Delta v\|^2 + \frac{a^2}{4\varepsilon_3} \|\nabla u\|^2.
\]
(26)

So, by using the Sobolev inequality in (25) we find
\[
\frac{d}{dt} E_2(t) \leq \frac{\mu_1^2}{2\varepsilon_1} \|\nabla (u_2)\|^2 + \frac{a^2 d_2}{2\varepsilon_2} \|\Delta v\|^2 + \frac{a^2}{4\varepsilon_3} \|\nabla u\|^2,
\]
(27)
where \( d_2 \) is a positive constant in the Sobolev inequality. Inequality (26) implies
\[
\frac{d}{dt} E_2(t) \leq \frac{\mu_1^2}{2\varepsilon_1} \|\nabla (u_2)\|^2 + M_2 E_2(t),
\]
(28)
where \( M_2 = a^2 \max \left\{ 1, \frac{d_z}{e_z y}, \frac{1}{2e_3} \right\} \). If we choose
\[ e_1 = \frac{\mu_1}{2}, \]
then we can write
\[
\frac{d}{dt} E_2(t) - M_2 E_2(t) \leq \frac{\mu_2^2}{\mu_1} \left\| \nabla (u_2) \right\|^2. \tag{29}
\]
Finally, Gronwall’s inequality gives
\[ E_2(t) \leq \mu^2 A_1(t), \]
where
\[ A_1(t) = \frac{1}{\mu_1} e^{\mu_1 t} \int_0^t \left\| \nabla (u_2) \right\|^2 ds. \]
Hence the statement of the theorem holds and we have
\[ \left\| u_1 \right\|^2 + \left\| v_1 \right\|^2 + \left\| \Delta v \right\|^2 + \left\| \nabla u \right\|^2 \to 0 \]
as \( \mu \to 0 \).

Finally, we show that the solution of the problem (1)-(5) depends continuously on the coefficient \( h \). Assume that \( (u_1, v_1) \) is the solution of the problem
\[
\alpha (u_1)_x - \Delta u_1 - \mu \Delta u_1 + a \Delta v = 0 \quad x \in \Omega, \quad t > 0,
\beta (v_1)_x + \gamma \Delta^2 v_i + a \Delta u_1 - h_i \Delta (v_1)_i = 0 \quad x \in \Omega, \quad t > 0,
\]
\[
(u_1(x, 0), v_1(x, 0)) = (u_0(x), v_0(x)) \quad x \in \Omega,
((u_1)_x(x, 0), (v_1)_x(x, 0)) = (u_1(x), v_1(x)) \quad x \in \Omega,
\]
\[ u_1 = v_1 = \frac{\partial v_1}{\partial v} = 0, \quad x \in \partial \Omega, \quad t > 0,
\]
and \( (u_2, v_2) \) is the solution of the following problem
\[
\alpha (u_2)_x - \Delta u_2 - \mu \Delta u_2 + a \Delta v_2 = 0 \quad x \in \Omega, \quad t > 0,
\beta (v_2)_x + \gamma \Delta^2 v_2 + a \Delta u_2 - h_2 \Delta (v_2)_i = 0 \quad x \in \Omega, \quad t > 0,
\]
\[
(u_2(x, 0), v_2(x, 0)) = (u_0(x), v_0(x)) \quad x \in \Omega,
((u_2)_x(x, 0), (v_2)_x(x, 0)) = (u_2(x), v_2(x)) \quad x \in \Omega,
\]
\[ u_2 = v_2 = \frac{\partial v_2}{\partial v} = 0, \quad x \in \partial \Omega, \quad t > 0.
\]
Let \( u = u_1 - u_2, \ v = v_1 - v_2 \) and \( h = h_1 - h_2 \). Then \( (u, v) \) satisfies the problem
\[
\alpha u_x - \Delta u - \mu \Delta u + a \Delta v = 0 \quad x \in \Omega, \quad t > 0, \tag{30}
\beta v_x + \gamma \Delta^2 v + a \Delta u - h \Delta (v)_i = 0 \quad x \in \Omega, \quad t > 0,
\]
\[
(u(x, 0), v(x, 0)) = (0, 0) \quad x \in \Omega, \tag{32}
(u_i(x, 0), v_i(x, 0)) = (0, 0) \quad x \in \Omega, \tag{33}
\]
\[ u = v = \frac{\partial v}{\partial v} = 0, \quad x \in \partial \Omega, \quad t > 0. \tag{34}
\]

The last result of this section is the following theorem.

\textbf{Theorem 3.} Let \( u \) and \( v \) be the solutions of the problem (30)-(34). Then the following inequality holds:
\[
\left\| u \right\|^2 + \left\| v \right\|^2 + \left\| \Delta v \right\|^2 + \left\| \nabla u \right\|^2 \leq \left( h_1 - h_2 \right)^2 A_2(t), \quad \forall t > 0. \tag{35}
\]

\textbf{Proof.} Multiplying (30) and (31) by \( u_i \) and \( v_i \) in \( L^2(\Omega) \), respectively and adding the obtained relations, we obtain
\[
\frac{d}{dt} E_2(t) + \mu \left\| \nabla u \right\|^2 + h_i \left\| \nabla v \right\|^2 + h \left( \nabla (v)_x, \nabla v \right) + a \left( \nabla u, \nabla v \right) - a \left( \nabla v, \nabla u \right) = 0. \tag{36}
\]

Similar to the proof of Theorem 2, we obtain the following inequality from (36):
\[
\frac{d}{dt} E_2(t) \leq \frac{h_1}{h_2} \left\| \nabla (v)_x \right\|^2 + M_2 E_2(t), \tag{37}
\]
and so, this completes the proof of Theorem 3.

\textbf{REFERENCES}


