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E^3 de Birinci Asli Yön Eğrisiyle Elde Edilen A- net Regle Yüzeyler

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ÖZ

Bu makalede, 3- boyutlu Öklid uzayında, birinci asli yön eğrisiyle elde edilen A- net regle yüzeylerin yeni bir parametrizasyonu verildi. Daha sonra, bu yüzeylerin ortalama eğriliği ve Gauss eğriliği elde edildi. Sonuç olarak, A- net yüzeylerin ortalama eğriliğe göre karakterizasyonu elde edildi.

Anahtar kelimeler: Regle yüzey, A- net yüzey, birinci asli yön eğrisi, ortalama eğrilik, Gauss eğrilik.

A-Net Ruled Surface Which Generated By First Principle Direction Curve In E^3

ABSTRACT

In this paper, we give a new parametrization of A- net surface which generated by first principle direction curve Euclidean 3- space E^3. Then, we obtain mean curvature and Gaussian curvature of this surface. Finally, we characterize A- net surface according to mean curvature in E^3.

Keywords: Ruled surface, A- net surface, first principle direction curve, mean curvature, Gaussian curvature.

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1. INTRODUCTION

The Frenet frame is generally known an orthonormal vector frame for curves. But, it does not always meet the needs of curve characterizations. In [1, 2, 3], there is another moving frame and the Sabban frame, respectively. Then, they gave some new characterizations of the C-slant helix and demonstrate that a curve of C-constant precession is a C-slant helix.

Ruled surfaces are one of the most important topics of differential geometry. The surfaces were found by Gaspard Monge, who was a French mathematician and inventor of descriptive geometry. Besides, these surfaces have the most important position of the study of one parameter motions. In [4, 5], ruled surfaces studied in Sol Space Sol^3. In [6] and [7] studied surfaces which have constant curvature and developable surfaces.

If an isometric representment between two surfaces conserve the principal curvatures of these surface, then, these surfaces are called Bonnet surfaces. In [8, 9], the Bonnet problem of detection the surfaces in three dimensional Euclidean space E^3 which can accept at least one nontrivial isometry that conserves principal curvatures is considered. This problem thought-out locally and extend to the general case. Then, in order to find a Bonnet surface a method is obtained. So, such a A-net on a surface, when this net is parametrized, the conditions $E = G$, $F = 0$, $h_{12} = c = const. \neq 0$ are satisfied, is named an A-net, where E, F, G are the coefficients of the first fundamental form of the surface and h_{11}, h_{12}, h_{22} are the coefficients of the second fundamental form. Then, in [10, 11], it is considered the Bonnet ruled surfaces which approve only one non-trivial isometry that preserves the principal curvatures, then, the definition of the A- net surface is given and determined the Bonnet ruled surfaces whose generators and orthogonal trajectories form a special net called an A-net. Using the ruled Bonnet surfaces the question of finding some pairs of orthogonal ruled surfaces is considered in [11], then, it is exemplified that only one pair of orthogonal ruled surfaces can be deduced. The problem of detection the Bonnet hypersurfaces in R^{n+1}, for $n > 1$ is disscused in [12].

In this paper, we study A- net surface which generated by first principle direction curve according to alternative moving frame Euclidean 3- space E^3. Then, we give some characterizations of this surface.

2. PRELIMINARIES

Let ϕ be an oriented surface in three dimensional Euclidean space E^3 and γ be a curve lying on the surface ϕ. Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet-Serret frame along the curve γ in the space E^3. For an arbitrary curve γ the Frenet--Serret formula is

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N}, \\ \mathbf{N}' &= -\kappa \mathbf{T} + \boldsymbol{\tau} \mathbf{B}, \\ \mathbf{B}' &= -\tau \mathbf{N}. \end{aligned}$$

where κ and τ first and second curvature in the space E^3,

Definition 2.1. Let $\gamma(s)$ be a regular unit-speed curve in terms of $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. The integral curves of $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are named the tangent direction curve, principal direction curve and binormal direction curve of $\gamma(s)$, respectively [11] .

The principal direction curve of $\gamma(s)$, $\alpha = \int \mathbf{N}(s) ds$ has a new frame as,

$$\begin{aligned} \mathbf{T}_1 &= \mathbf{N}, \\ \mathbf{N}_1 &= \frac{\mathbf{N}'}{\|\mathbf{N}\|} = \frac{-\kappa \mathbf{T} + \boldsymbol{\tau} \mathbf{B}}{\sqrt{\kappa^2 + \tau^2}}, \\ \mathbf{B}_1 &= \mathbf{T}_1 \times \mathbf{N}_1 = \frac{\boldsymbol{\tau} \mathbf{T} + \boldsymbol{\kappa} \mathbf{B}}{\sqrt{\kappa^2 + \tau^2}}. \end{aligned}$$

where the tangent vector and the binormal vector of α are the principle normal vector and the unit Darboux vector of $\gamma(s)$, respectively. Also curvatures of α are

$$\kappa_1 = \sqrt{\kappa^2 + \tau^2}, \tau_1 = \sigma \kappa_1$$

where σ is the geodesic curvature , which measures how far the curve is from being geodesic, of $\mathbf{N}_1$. The first principle direction curve,

$$\alpha = \int \mathbf{N}(\mathbf{s}) ds$$

with the frame

$$\{\mathbf{T}_1 = \mathbf{N}, \mathbf{N}_1 = \frac{\mathbf{N}'}{\|\mathbf{N}\|}, \mathbf{B}_1 = \mathbf{T}_1 \times \mathbf{N}_1\}.$$

The Frenet equations are satisfied

$$\mathbf{T}_1' = \kappa_1 \mathbf{N}_1,$$
$$\mathbf{N}_1' = -\kappa_1 \mathbf{T}_1 + \boldsymbol{\tau}_1 \mathbf{B}_1,$$
$$\mathbf{B}_1' = -\tau_1 \mathbf{N}_1.$$

Components of the first fundamental form of a surface $\Omega(s,v)$ are $E = \langle \Omega_s, \Omega_s \rangle$, $F = \langle \Omega_s, \Omega_v \rangle$ and $G = \langle \Omega_v, \Omega_v \rangle$. Components of the second fundamental form of a surface $\Omega(s,v)$ are $h_{11} = \langle \Omega_{ss}, \mathbf{U} \rangle$, $h_{12} = \langle \Omega_{sv}, \mathbf{U} \rangle$ and $h_{22} = \langle \Omega_{vv}, \mathbf{U} \rangle$ where $\mathbf{U}$ is the unit normal vector field of the surface $\Omega(s,v)$.

Mean curvature and Gaussian curvature are fundamental to the study of the geometry of surfaces. H and K are defined in terms of the components of the first and second fundamental forms

$$H = \frac{Eh_{22} - 2Fh_{12} + Gh_{11}}{2(EG - F^2)},$$

$$K = \frac{h_{11}h_{22} - h_{12}^{\,2}}{EG - F^2}.$$

When the mean curvature of the surface is zero, then this surface called minimal surface [13].

A ruled surface in E^3 is the map $\phi_{(\gamma,\delta)} : I \times R \to E^3$ defined by

$$\phi_{(\gamma,\delta)}(s,v) = \gamma(s) + v\delta(s).$$

We call the $\gamma(s)$ and $\delta(s)$ as base curve and the director curve. The straight lines $u \to \gamma(s) + u\delta(s)$ are called rulings of $\phi_{(\gamma,\delta)}$.

Definition 2.2. A surface formed by a singly infinite system of straight lines is called a ruled surface. Let $\gamma : I \to E^3$ be a unit speed curve. We define the following ruled surface

$$X_{(\gamma,\delta)}(s,v) = \gamma(s) + v\mathbf{T}(s)$$

where $\mathbf{T}(s)$ is unit tangent vector field of γ.

3. A-NET RULED SURFACE GENERATED BY FIRST PRINCIPLE DIRECTION CURVE E^3

In this section, we characterize A-net surfaces in Euclidean 3- space E^3. Then, we obtain constant mean curvature and Gaussian curvature of this surface. A-net on a surface is defined following conditions

$$E = G, F = 0, h_{12} = constant \neq 0,$$

where E, F, G are the coefficients of the first fundamental form of the surface and h_{11}, h_{12}, h_{22} are the coefficients of the second fundamental form.

Theorem 3.1. Let

$$X(s,v) = \alpha(s) + f(v)\mathbf{N}_1(s) \tag{1}$$

in E^3. If the surface $X(s,v)$ is a A-net surface, then

$$\kappa_1 = -\tau_1, \kappa_1 = e^{-s}.c_1 \tag{2}$$

and

$$f'^2 = 1 - 2fe^{-s}c_1 + 2f^2e^{2s}c_1^2.$$

where σ is the geodesic curvature of $\mathbf{N}_1$ and c_1 is constant of integration.

Proof. If we take derivatives of the surface, which is given with the parametrization (1), we have

$$X_s = (1 - f\kappa_1)\mathbf{T}_1 + f\tau_1\mathbf{B}_1, \tag{3}$$
$$X_v = f'\mathbf{N}_1.$$

Then, components of the first fundamental form of the surface are

$$E = (1 - f\kappa_1)^2 + f^2\tau_1^2,$$
$$F = 0, \tag{4}$$
$$G = f'^2.$$

The unit normal vector field of the surface is

$$\mathbf{U} = \frac{1}{\sqrt{(1 - f\kappa_1)^2 + f^2\tau_1^2}}\left(-\tau_1 f\mathbf{T}_1 + (1 - f\kappa_1)\mathbf{B}_1\right). \tag{5}$$

Second derivatives of the surface are

$$X_{ss} = -f\kappa_1'\mathbf{T}_1 + ((1 - f\kappa_1)\kappa_1 - \tau_1^2 f)\mathbf{N}_1 + f\tau_1'\mathbf{B}_1$$
$$X_{sv} = f'(-\kappa_1\mathbf{T}_1 + \tau_1\mathbf{B}_1)$$
$$X_{vv} = f''\mathbf{N}_1.$$

Then, components of the second fundamental form of the surface are

$$h_{11} = \frac{f^2\kappa_1'\tau_1 + f\tau_1'(1 - f\kappa_1)}{\sqrt{(1 - f\kappa_1)^2 + f^2\tau_1^2}}, \tag{6}$$

$$h_{12} = \frac{f' \tau_1}{\sqrt{(1-f\kappa_1)^2 + f^2\tau_1^2}}, \qquad (7)$$

$$h_{22} = 0. \qquad (8)$$

From the definition of the A- net surface and equations (4) and (6), (7) and (8), we have

$$\frac{f' \tau_1}{\sqrt{(1-f\kappa_1)^2 + f^2\tau_1^2}} = const. \qquad (9)$$

Derivative of the equation (9) according to s and v, we obtain following differential equations

$$\kappa_1(f\kappa_1 - 1) + v\tau_1^2 = 0, \qquad (10)$$

$$\tau_1(f\kappa_1 - 1) - v\tau_1\kappa_1' = 0. \qquad (11)$$

From (10), (11) and $\tau_1 = \sigma\kappa_1$, we have

$$\kappa_1\{(f\kappa_1 - 1)(1+\sigma) + v\sigma\kappa_1(\sigma\kappa_1 - \kappa_1')\} = 0. \qquad (12)$$

Because of $\kappa_1 \neq 0$ and $\sigma \neq 0$, we have (2). In these conditions, because of $E = G$, we have

$$f'^2 = 1 - 2fe^{-s}c_1 + 2f^2e^{2s}c_1^2.$$

Corollary 3.1. Let $X(s,v)$ be a surface which is given by equation (8) in E^3. If

$$\kappa_1 = const.$$

and $\sigma = const.$, (13)

then, $X(s,v)$ is a minimal surface.

Proof. From equations (4) and (6)- (8), the equation of mean curvature is

$$H = \frac{f(f\kappa_1'\tau_1 + \tau_1'(1-f\kappa_1))}{2((1-f\kappa_1)^2 + f^2\tau_1^2)^{3/2}}. \qquad (14)$$

So, if the surface $X(s,v)$ is minimal, then

$$\kappa_1 = const. \text{and } \sigma = const.$$

Corollary 3.2. If $X(s,v)$ is a A- net minimal surface, then

$$s = \ln|c_2|$$

and

$$f'^2 = 1 - 2fc_3 + 2f^2c_3^2$$

where c_2 and c_3 are constant.

Proof. The proof obtains from equations (2), (13).

Corollary 3.3. Let $X(s,v)$ be a surface which is given by equation (1) in E^3. Then, the Gaussian curvature of the surface

$$K = -\frac{\tau_1^2}{((1-f\kappa_1)^2 + f^2\tau_1^2)^2}.$$

Proof. The proof is obtained from equations (4), (6)-(8).

Example 3.1. Let $\gamma(s)$ be a curve which has the principal normal vector field

$$\mathbf{N}(s) = (a\cos(\frac{s}{\sqrt{a^2+b^2}}), a\sin(\frac{s}{\sqrt{a^2+b^2}}), b\frac{s}{\sqrt{a^2+b^2}}).$$

Then, first principle direction curve is

$$\alpha(s) = (a\sqrt{a^2+b^2}\sin(\frac{s}{\sqrt{a^2+b^2}}), -a\sqrt{a^2+b^2}\cos(\frac{s}{\sqrt{a^2+b^2}}), \frac{s^2}{2\sqrt{a^2+b^2}}).$$

The ruled surface which generated by first principle direction curve is

$$X(u,v) = (a\sqrt{a^2+b^2}\sin(\frac{s}{\sqrt{a^2+b^2}}) + a\sin v\cos(\frac{s}{\sqrt{a^2+b^2}}),$$

$$-a\sqrt{a^2+b^2}\cos(\frac{s}{\sqrt{a^2+b^2}}) + a\sin v\sin(\frac{s}{\sqrt{a^2+b^2}}),$$

$$\frac{s^2}{2\sqrt{a^2+b^2}} + b\sin v\frac{s}{\sqrt{a^2+b^2}})$$

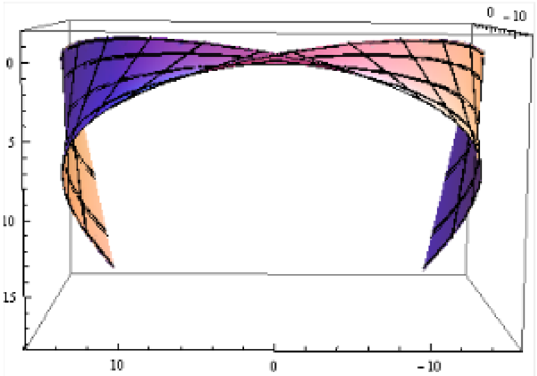

Figure 1. The surface graph of the ruled surface which is generated by first principle direction curve for a=3 and b=4

Theorem 3.2. Let

$$\psi(s,v)=\alpha(s)+f(v)\mathbf{B}_1(s) \tag{15}$$

in E^3. If the surface $\psi(s,v)$ is a A-net surface, then following differential equation holds;

$$f(v)=e^{\sqrt{2cv}}c_4+e^{-\sqrt{2cv}}c_5 \tag{16}$$

where c_4 and c_5 are constant.

Proof. If we take derivatives of the surface, which is given with the parametrization (15), we have

$$\psi_s=\mathbf{T}_1-f\tau_1\mathbf{N}_1, \tag{17}$$

$$\psi_v=f'\mathbf{B}_1.$$

Then, components of the first fundamental form of the surface are

$$E=1+f^2\tau_1^2,$$
$$F=0, \tag{18}$$
$$G=f'^2.$$

The unit normal vector field of the surface is

$$\mathbf{U}=-\frac{1}{\sqrt{1+f^2\tau_1^2}}\left(\tau_1 f\mathbf{T}_1+\mathbf{N}_1\right). \tag{19}$$

Second derivatives of the surface are

$$\psi_{ss}=-f\kappa_1\tau_1\mathbf{T}_1+(\kappa_1-f\tau_1')\mathbf{N}_1-f\tau_1^2\mathbf{B}_1$$

$$\psi_{sv}=-f'\tau_1\mathbf{N}_1$$

$$\psi_{vv}=f''\mathbf{B}_1.$$

Then, components of the second fundamental form of the surface are

$$h_{11}=-\frac{f^2\kappa_1\tau_1^2+\kappa_1-f\tau_1'}{\sqrt{1+f^2\tau_1^2}}, \tag{20}$$

$$h_{12}=\frac{f'\tau_1}{\sqrt{1+f^2\tau_1^2}}, \tag{21}$$

$$h_{22}=0. \tag{22}$$

From the definition of the A- net surface, from equation (21), we have

$$\frac{f'\tau_1}{\sqrt{(1-f\kappa_1)^2+f^2\tau_1^2}}=\zeta, \tag{23}$$

where $\zeta=const\neq 0$. Derivative of the equation (23) according to s and v, we obtain following differential equations

$$f'\tau_1'=0, \tag{24}$$

$$f''(1+\tau^2f^2)-\tau^2ff'^2=0. \tag{25}$$

Because of $f'\neq 0$ and $\tau_1\neq 0,$ from (24), we have

$$\tau_1=const.=c_5. \tag{26}$$

Then, from (25) and (26),

$$(f''-c_4f)(1+c_4^2f^2)=0.$$

Because of $c_4^2f^2\neq 1,$

$$f(v)=e^{\sqrt{2cv}}c_4+e^{-\sqrt{2cv}}c_5.$$

Corollary 3.4. Let $\psi(s,v)$ be a surface which is given by equation (15) in E^3. Then, the mean curvature of the surface $\psi(s,v)$

$$H=-\frac{f^2\kappa_1\tau_1^2+\kappa_1-f\tau_1'}{(1+f^2\tau_1^2)^{3/2}}.$$

Proof. It is obviousliy from equations (18), (20)- (22).

Corollary 3.5. Let $\psi(s,v)$ be a surface which is given by equation (15) in E^3. Then, the Gaussian curvature of the surface

$$K=-\frac{\tau_1^2}{(1+f^2\tau_1^2)^2}.$$

Proof. It is obviousliy from equations (18), (20)- (22).

Corollary 3.6. *Let*

$$\xi(s,v)=\alpha(s)+f(v)\mathbf{T}_1(s)$$

in E^3. $\xi(s,v)$ isn't a A- net surface.

Proof. With similar computations in Theorem 3.1 and Theorem 3.5, coefficients of the first fundamental form

$$E = 1 + f^2\kappa_1^2,$$
$$F = f',$$
$$G = f'^2.$$

and components of the second fundamental form

$$h_{11} = -f\kappa_1\tau_1,$$
$$h_{12} = 0,$$
$$h_{22} = 0.$$

Since $h_{12} = 0$, $\xi(s,v)$ isn't a A- net surface.

Corollary 3.7. *Let*

$$Y(s,v) = \alpha(s) + vc\mathbf{T}_1(s),$$
$$Z(s,v) = \alpha(s) + vc\mathbf{N}_1(s)$$

and

$$\mathrm{T}(s,v) = \alpha(s) + vc\mathbf{B}_1(s)$$

be ruled surfaces in E^3. $Y(s,v)$, $Z(s,v)$ or $\mathrm{T}(s,v)$ aren't a A- net surface.

Proof. Because of second components of the second fundamental formas of these surfaces ($h_{12} = 0$), $Y(s,v)$, $Z(s,v)$ or $\mathrm{T}(s,v)$ aren't a A- net surface.

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