Some sums related to the terms of generalized Fibonacci autocorrelation sequences \( \left\{ a_{k,n}(\tau) \right\}_\tau^\infty \)

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**ABSTRACT**

In this paper, we give the terms of the generalized Fibonacci autocorrelation sequences \( \left\{ a_{k,n}(\tau) \right\}_\tau^\infty \) defined as
\[
a_{k,n}(\tau) := a_n(U_k, \tau)
\]
and some interesting sums involving terms of these sequences for an odd integer number \( k \) and nonnegative integers \( \tau, n \).

**Keywords:** Fibonacci numbers, generalized Fibonacci autocorrelation sequences, sums

\[
\left\{ a_{k,n}(\tau) \right\}_\tau^\infty \text{ geneleştirilmiş Fibonacci otokorelasyon dizilerinin terimlerini içeren bazı bağıntılar}
\]

**ÖZ**

Bu makalede, \( k \) tek sayı ve \( \tau, n \) negatif olmayan tam sayı olmak üzere
\[
a_{k,n}(\tau) := a_n(U_k, \tau)
\]
terimlerine sahip \( \left\{ a_{k,n}(\tau) \right\}_\tau^\infty \) geneleştirilmiş Fibonacci otokorelasyon dizileri ve bu dizilerin terimlerini içeren bazı toplamlar verildi.

**AnahtarKelimeler:** Fibonacci sayıları, geneleştirilmiş Fibonacci otokorelasyon dizileri, toplamlar

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1. **GİRİŞ (INTRODUCTION)**

For \( a, b, p, q \in \mathbb{R} \), the second order sequence \( \{W_n(a, b; p, q)\} \) is defined for \( n > 0 \) by

\[
W_{n+1}(a, b; p, q) = pW_n(a, b; p, q) - qW_{n-1}(a, b; p, q)
\]

in which \( W_0(a, b; p, q) = a \), \( W_1(a, b; p, q) = b \).

When \( q = -1 \), \( W_n(0, 1; p, -1) = U_n \) and \( W_n(2, p; p, -1) = V_n \). When \( p = 1 \), \( U_n = F_n \) (n-th Fibonacci number) and \( V_n = L_n \) (n-th Lucas number).

If \( \alpha \) and \( \beta \) are the roots of equation \( x^2 - px - 1 = 0 \) the Binet formulas of the sequences \( \{U_n\} \) and \( \{V_n\} \) have the forms

\[
U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,
\]

respectively.

E. Kilç and P. Stanica [1], derived the following recurrence relations for the sequences \( \{U_{k_n}\} \) and \( \{V_{k_n}\} \) for \( k \geq 0 \), \( n > 0 \).

\[
U_{k(n+1)} = V_kU_{k_n} + (-1)^{k+1}U_{k(n-1)}
\]

and

\[
V_{k(n+1)} = V_kV_{k_n} + (-1)^{k+1}V_{k(n-1)},
\]

where the initial conditions of the sequences \( \{U_{k_n}\} \) and \( \{V_{k_n}\} \) are \( 0 \), \( U_k \) and \( 2 \), \( V_k \) respectively. The Binet formulas of the sequences \( \{U_{k_n}\} \) and \( \{V_{k_n}\} \) are given by

\[
U_{k_n} = \frac{\alpha^{k_n} - \beta^{k_n}}{\alpha - \beta} \quad \text{and} \quad V_{k_n} = \alpha^{k_n} + \beta^{k_n},
\]

respectively.

P. Filipponi and H.T. Freitag [2] defined the terms \( a_n(S_i, \tau) \) of the autocorrelation sequences of any sequence \( \{S_i\}_{i=0}^{\infty} \) as

\[
a_n(S_i, \tau) := \sum_{i=0}^{n} S_{i+S_{i+\tau}}, \quad (0 \leq \tau \leq n), \tag{1}
\]

where the subscript \( i + \tau \) must be considered as reduced modulo \( n + 1 \) and \( n \) are nonnegative integers. It is clearly that autocorrelation sequences differ from the definition of cyclic autocorrelation function for periodic sequences with period \( n+1 \) [3].

For positive integer number \( \tau \), the authors gave

\[
a_n(S_i, \tau) = a_n(S_i, n - \tau + 1)
\]

and

\[
a_n(S_i, \tau) = \sum_{i=0}^{n-\tau} S_{i+S_{i+\tau}} + \sum_{i=0}^{\tau-1} S_{i+n-\tau+i}.
\]

The terms of the Fibonacci autocorrelation sequences \( \{a_{k,n}(\tau)\}_{\tau}^{\infty} \) were defined as

\[
a_n(\tau) := a_n(F, \tau)
\]

and they obtained some sums involving the terms \( a_n(\tau) \) as follows:

\[
\sum_{i=0}^{n} a_n(i) = (F_{n+2} - 1)^2,
\]

\[
10 \sum_{i=0}^{n} a_n(i)
\]

\[
= \begin{cases} 2L_{3n+2} - 5F_{2n+2} + L_{n+1}, & \text{if } n \text{ is even} \nonumber \\ 2L_{3n+2} - L_{n+1}(5F_{n+1} - 1), & \text{if } n \text{ is odd} \end{cases}
\]

Inspiring by studies in [2], we consider subsequence \( \{S_{k_n}\}_{i=0}^{\infty} \) of the autocorrelation sequences of subsequence \( \{S_i\}_{i=0}^{\infty} \) defined as

\[
a_n(S_{k_n}, \tau) := \sum_{i=0}^{n} S_{k_n+S_{k_n(\tau+i)}}, \quad (0 \leq \tau \leq n), \tag{2}
\]

where the subscript \( k_n + \tau \) must be considered as reduced modulo \( n + 1 \). It can clearly be seen that
\[ a_n(S_{k}, \tau) = a_n(S_{k}, n - \tau + 1) \]  \hspace{1cm} (3) \\
and \\
\[ \sum_{i=0}^{n} S_{ki}S_{k(i+r)} = \sum_{i=0}^{n-\tau} S_{ki}S_{k(i+r)} + \sum_{i=0}^{r-1} S_{k(i+n-\tau+1)}S_{ki}, \] 

where \( \tau \) is positive integer number.

For example, for \( n = 6 \), \( k = 5 \) and \( \tau = 3 \) in (3), \\
\[ a_{6}(S_{5}, 3) = S_{0}S_{15} + S_{2}S_{20} + S_{10}S_{25} + S_{15}S_{30} + S_{20}S_{0} + S_{25}S_{5} + S_{30}S_{10} = a_{6}(S_{5}, 4). \]

In this paper, taking generalized Fibonacci subsequence \( \{U_{ki}\}_{0}^{\infty} \) instead of subsequence \( \{S_{ki}\}_{0}^{\infty} \) in (2), we write the terms of the generalized Fibonacci autocorrelation sequences \( \{a_{k,n}(\tau)\}_{\infty} \) as \\
\[ a_{k,n}(\tau) = \sum_{i=0}^{n} U_{ki}U_{k(i+r)} \]

and obtain some sums involving the numbers \( a_{k,n}(\tau) \), where an odd integer \( k \) and nonnegative integers \( \tau \), \( n \). Throughout this paper, we will take \( \{W_{n}\} \) instead of \( \{W_{n}(a,b;p,q)\} \).

The following Fibonacci identities and sums in [4] will be used widely throughout the proofs of Theorems:

\[ V_{k(n+m)} + V_{k(m-n)} = \begin{cases} V_{km}V_{kn}, & \text{if } n \text{ is even} \\ \Delta U_{kn}U_{km}, & \text{if } n \text{ is odd} \end{cases} \]  \hspace{1cm} (4) \\
\[ V_{k(m+n)} - V_{k(m-n)} = \begin{cases} \Delta U_{km}U_{kn}, & \text{if } n \text{ is even} \\ V_{km}V_{kn}, & \text{if } n \text{ is odd} \end{cases} \]  \hspace{1cm} (5) \\
\[ U_{k(m+n)} + U_{k(m-n)} = \begin{cases} U_{km}V_{kn}, & \text{if } n \text{ is even} \\ U_{km}V_{kn}, & \text{if } n \text{ is odd} \end{cases} \]  \hspace{1cm} (6) \\
\[ U_{k(m+n)} - U_{k(m-n)} = \begin{cases} V_{km}U_{kn}, & \text{if } n \text{ is even} \\ U_{km}V_{kn}, & \text{if } n \text{ is odd} \end{cases} \]  \hspace{1cm} (7)

\[ \sum_{i=r}^{n} W_{k(i+r)} = \left[ W_{k(r+c+d)} - W_{k(c(n+1)+d)} - (-1)^{c} W_{k(c(r+1)+d)} \right] \left[ 1 + V_{kc} + (-1)^{c} \right] \]  \hspace{1cm} (8) \\
\[ + (-1)^{c} W_{k(c(n+r)+d)} \]  \hspace{1cm} (9) \\
\[ + \frac{(-1)^{c}}{1 + V_{kc} + (-1)^{c}}, \]  \hspace{1cm} (10)

and \\
\[ \sum_{i=r}^{n} (-1)^{c} iW_{k(i+r)} = \left[ \left( r + 2(r-1)(-1)^{c} \right) \right] \left[ 1 + V_{kc} + (-1)^{c} \right] \]  \hspace{1cm} (11)


2. SOME IDENTITIES INVOLVING THE TERMS \( a_{k,n}(\tau) \) \( (a_{k,n}(\tau) \) TERİMLERİNDER \( İÇEREN BAZI ÖZELLİKLERİ)
In this section, we will give closed-form expressions for terms of the generalized Fibonacci autocorrelation sequences \( \{a_{k,n}(\tau)\}_{\tau}^\infty \). Now, we give auxiliary Lemma before the proof of main Theorems.

**Lemma 2.1.** Let \( k \) be an odd integer number.

For even \( \tau \),

\[
V_k a_{k,n}(\tau) = \begin{cases} 
U_{k(n+1)} U_{k(n-\tau+1)} + U_{k,0} U_{k,\tau}, & \text{if } n \text{ is even}, \\
U_{k,n} \left( U_{k(n-\tau+1)} + U_{k,\tau} \right), & \text{if } n \text{ is odd}
\end{cases}
\]

and for odd \( \tau \),

\[
V_k a_{k,n}(\tau) = \begin{cases} 
U_{k,n} U_{k(n-\tau+1)} + U_{k(n+1)} U_{k(\tau-1)}, & \text{if } n \text{ is even}, \\
U_{k(n+1)} \left( U_{k(n-\tau)} + U_{k(\tau-1)} \right), & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof.** Let \( n \) and \( \tau \) be even integers. Using Binet formula of generalized Fibonacci sequence \( \{U_{kn}\} \), we write

\[
a_{k,n}(\tau) = \sum_{i=0}^{n-\tau} U_{k,i} U_{k(n+1)-\tau+i} + \sum_{i=0}^{\tau-1} U_{k,i} U_{k(n+1)-\tau+i} 
\]

\[
= \frac{1}{\Delta} \left\{ \sum_{i=0}^{n-\tau} \left( \alpha^i \beta^{(2i+\tau)} + \beta^i \alpha^{(2i+\tau)} - \beta^{2i} \alpha^{(i+\tau)} - \alpha^i \beta^{(i+\tau)} \right) 
\right.
\]

\[
+ \sum_{i=0}^{\tau-1} \left( \alpha^{2i+n-\tau+1} + \beta^{2i+n-\tau+1} - \beta^{2i} \alpha^{(i+n-\tau+1)} - \alpha^i \beta^{(i+n-\tau+1)} \right) \right\}
\]

\[
= \frac{1}{\Delta} \left\{ \sum_{i=0}^{n-\tau} \left( V_{k(2i+\tau)} - (-1)^i V_{k(\tau+1)} \right) 
\right.
\]

\[
+ \sum_{i=0}^{\tau-1} \left( V_{k(2i+n-\tau+1)} - (-1)^i V_{k(n+1)} \right) \right\}
\]

From (8) and the sums

\[
\sum_{i=0}^{n-\tau} (-1)^i = \begin{cases} 
1, & \text{if } n, \tau \text{ are same parities}, \\
0, & \text{if } n, \tau \text{ are different parities}
\end{cases} 
\]

\[
\sum_{i=0}^{\tau-1} (-1)^i = \begin{cases} 
1, & \text{if } \tau \text{ is odd,} \\
0, & \text{if } \tau \text{ is even}
\end{cases}
\]

we write

\[
\Delta a_{k,n}(\tau) = \left( V_{k(2n-\tau+1)} - V_{k(\tau+1)} + V_{k(n+\tau)} - V_{k(n-\tau)} \right) / V_k .
\]

By (5), we have

\[
V_k a_{k,n}(\tau) = U_{k(n+1)} U_{k(n-\tau+1)} + U_{k,0} U_{k,\tau}.
\]

The other equalities are obtained similar to the proof. Thus we have the conclusion.

For example, for \( a = \tau = 0 \) and \( b = k = p = 1 \) in Lemma 2.1, it is clearly seen that \( a_{k,n}(0) = F_{n+1} F_n / [1] \).

Now, we will investigate some sums involving the terms \( a_{k,n}(\tau) \).

**Theorem 2.1.** Let \( k \) be an odd integer number. We have

\[
V_k \sum_{\tau=0}^{n} (-1)^{\tau} a_{k,n}(\tau) 
\]

\[
= \begin{cases} 
\left( U_{k(n+1)} - U_{k,n} + U_k \right)^2 / V_k, & \text{if } n \text{ is odd,} \\
U_{k(n+1)} U_{k,n}, & \text{if } n \text{ is even}
\end{cases}
\]

and

\[
V_k V_{3k} \sum_{\tau=0}^{n} (-1)^{\tau} a_{k,n-\tau}(i) 
\]

\[
= \begin{cases} 
\left( U_{k(n-1) + V_{k(2n+4)} - V_{k(2n+1)}} \right) / \Delta, & \text{if } n \text{ is odd,} \\
- V_{k(n-1)} + V_{3k} \left( V_k + V_{2k} \right) / \Delta V_k, & \text{if } n \text{ is even}
\end{cases}
\]

\[
= \begin{cases} 
\left( U_{k(n+2)} + V_{2k} (V_k - 2) \right) / \Delta V_k, & \text{if } n \text{ is odd,} \\
V_{k(n+2)} - V_{3k} (V_k - 2) / \Delta V_k, & \text{if } n \text{ is even}
\end{cases}
\]

**Proof.** For even number \( n \), observed that

\[
\sum_{\tau=0}^{n} (-1)^{\tau} a_{k,n}(\tau) = a_{k,n}(0) - a_{k,n}(1) + ... + a_{k,n}(n).
\]
From the equality \( a_{k,n}(i) = a_{k,n}(n-i+1) \) and Lemma 2.1, we get

\[
\sum_{i=0}^{n} (-1)^{i} a_{k,n}(i) = a_{k,n}(0) - a_{k,n}(1) + \ldots - a_{k,n}(n-1) + a_{k,n}(n) = a_{k,n}(0) - \sum_{r=1}^{(n-1)/2} a_{k,n}(2\tau) - \sum_{r=1}^{(n+1)/2} a_{k,n}(2\tau-1).
\]

For odd number \( n \), we write

\[
\sum_{i=0}^{n} (-1)^{i} a_{k,n}(i) = a_{k,n}(0) + \sum_{r=1}^{(n-1)/2} a_{k,n}(2\tau) = \sum_{i=0}^{n} (-1)^{i} a_{k,n}(i) = a_{k,n}(0) + \sum_{r=1}^{(n+1)/2} a_{k,n}(2\tau-1).
\]

Using the equality \( U_{k,n}^2 - U_{k(n-1),k(n+1)} = U_k^2 \) in [5], (7), (8) and Lemma 2.1, we have the claimed result. The remaining formulas are similarly proven.

**Theorem 2.2.** Let \( k \) be an odd integer number. We have

\[
V_k^2 \sum_{i=0}^{n} i a_{k,n}(i) = (n+1) \times \begin{cases} (U_{k(n+1)} + U_{kn})(U_{kn} - U_k), & \text{if } n \text{ is odd} \\ U_{k(n+1)}(U_{kn} + U_{k(n-1)} - U_k) - U_{kn} U_k, & \text{if } n \text{ is even} \end{cases}
\]

and

\[
V_k^2 \sum_{i=0}^{n} (-1)^{i} i a_{k,n}(i) = \begin{cases} (n+1)(U_{kn} - U_{k(n+1)})(U_{kn} - U_k), & \text{if } n \text{ is odd} \\ (n+1)(U_{kn} U_k - U_{k(n+1)} U_{k(n-1)}) + (n-1)U_{k(n+1)}(U_{kn} - U_k) + 4U_{kn}(U_{k(n+1)} - U_{kn})/V_k, & \text{if } n \text{ is even} \end{cases}
\]

**Proof.** For odd number \( n \), we write

\[
\sum_{i=0}^{n} a_{k,n}(i) = a_{k,n}(1) + 2a_{k,n}(2) + \ldots + na_{k,n}(n) = \sum_{i=0}^{(n-1)/2} 2ia_{k,n}(2i) + \sum_{i=0}^{(n-1)/2} (2i+1)a_{k,n}(2i+1).
\]

By Lemma 2.1, we have

\[
\sum_{i=0}^{n} i a_{k,n}(i) = \frac{1}{V_k} \left\{ U_{kn} \sum_{i=0}^{(n-1)/2} 2i(U_{k(n-2i+1)} + U_{2i}) + U_{k(n+1)} \sum_{i=0}^{(n+1)/2} (2i+1)(U_{k(n-2i-1)} + U_{2i}) \right\}.
\]

From (5), (7), (8) and (10), we get

\[
V_k^2 \sum_{i=0}^{n} i a_{k,n}(i) = (n+1)(U_{k(n+1)} + U_{kn})(U_{kn} - U_k)
\]

as claimed. Similarly, for even \( n \), the proof is clearly obtained. With the help of (11), the proof of the other result is given. Thus the proof is completed.

For example, taking \( a = 0 \) and \( b = k = 1 \) in (12), it is clearly seen that

\[
p^2 \sum_{i=0}^{n} (-1)^{i} a_{k,n}(i) = \begin{cases} (n+1)(U_{kn} - U_n)(1 - U_n), & \text{if } n \text{ is odd} \\ (n+1)(U_{kn} - U_{n+1} U_{n-1}) + (n-1)U_{n+1}(U_{n-1} - 1) + 4U_{n}(U_{n+1} - U_n)/p, & \text{if } n \text{ is even} \end{cases}
\]

**Theorem 2.3.** Let \( k \) be an odd integer number. We have

\[
\Delta U_{2k} \sum_{i=0}^{n} (-1)^{i(2i+1)} d_{k,n}(i) = \begin{cases} \Delta U_{k} U_{kn} U_{k(n+1)}, & \text{if } n \equiv 0 \pmod{4} \\ U_{k(n+1)} \times \begin{cases} (V_{k(n+1)} + V_{k(n-1)} + 2(V_k - V_{kn})), & \text{if } n \equiv 1 \pmod{4} \\ (V_k - 2)U_{k(n+1)} + (V_k + 2)U_k + 2V_k(U_{k(n+1)} - U_{kn}), & \text{if } n \equiv 2 \pmod{4} \\ V_k U_{kn}(V_{k(n+1)} - 2), & \text{if } n \equiv 3 \pmod{4} \end{cases} \end{cases}
\]
\[ \Delta V_k U_{3k} \sum_{i=0}^{n} (-1)^{(i+1)} a_{k,n-i}(i) \]

\[ = \begin{cases} 
U_{k(2n+4)} - U_{k(2n+1)} + U_{3k} (V_k + 1), & \text{if } n \text{ is odd} \\
(-1)^{\frac{n}{2}} (V_k + V_{2k}) U_{k(n-1)^2}, & \text{if } n \text{ is even}
\end{cases} \]

\[ + \frac{1}{\Delta U_{3k}} \left( V_k \left( U_{k(n+1)} + U_{kn} \right) (2 - V_{kn}) ight) \]

\[ \Delta U_k 2k \sum_{i=0}^{n} (-1)^{(i+1)} a_{k,n-i}(i) = \begin{cases} 
U_{k(2n+4)} - U_{k(2n+1)} + U_{3k} (V_k + V_{2k}) U_{k(n+1)^2} + U_{3k}, & \text{if } n \equiv 0(\text{mod } 4) \\
\Delta U_k \left( U_k^2 - U_{k(n+1)} U_{kn} - U_{kn}^2 \right), & \text{if } n \equiv 1(\text{mod } 4) \\
-\Delta U_k U_k U_{k(n+1)} U_{kn}, & \text{if } n \equiv 2(\text{mod } 4) \\
-\Delta U_k U_{kn} (V_k - 2), & \text{if } n \equiv 3(\text{mod } 4)
\end{cases} \]

**Proof.** For the second sum, the proof can be given. Let \( n \equiv 0(\text{mod } 4) \). Observed that

\[ \sum_{i=0}^{n} (-1)^{(i+1)} a_{k,n-i}(i) \]

\[ = (-1)^{\frac{1}{2}} a_{k,n} (0) + (-1)^{\frac{1}{2}} a_{k,n-1} + ... + (-1)^{\frac{n}{2}} a_{k,0} (n) \]

\[ = a_{k,n} (0) - a_{k,n-1} (1) + ... + a_{k,1} (n-1) + a_{k,0} (n) \]

\[ = a_{k,n} (0) + \sum_{i=1}^{\frac{n}{2}} a_{k,n-4i+1} (4i) + \sum_{i=1}^{\frac{n}{2}} a_{k,n-4i+3} (4i-3). \]

By (5) and Lemma 2.1, we get

\[ V_k \sum_{i=0}^{n} (-1)^{(i+1)} a_{k,n-i}(i) = U_{k(n+1)} U_{kn} \]

\[ + \frac{1}{\Delta U_{3k}} \left( V_{4ki} + U_{k(2n-12i+7)} \right) \]

From (4)-(6) and (8), we write

\[ \Delta V_k U_{3k} \sum_{i=0}^{n} (-1)^{(i+1)} a_{k,n-i}(i) \]

\[ = U_{k(n+1)} U_{kn} + \frac{1}{\Delta U_{3k}} \left( U_{k(n-1)} - U_{k(2n+1)} \right) \]

\[ - U_{k(2n-2)} + U_{k(n+2)} + \frac{1}{\Delta U_{k}} \left( U_{2k} - U_{k(n+2)} \right) \]

\[ + U_k - U_{k(n+1)} \]

\[ = \frac{U_{k(2n+4)} - U_{k(2n+1)} - (V_k + V_{2k}) U_{k(n+2)} + U_{3k}}{\Delta U_{3k}} \]

as claimed. For \( n = 2(\text{mod } 4) \), 

\[ \Delta V_k U_{3k} \sum_{i=0}^{n} (-1)^{(i+1)} a_{k,n-i}(i) \]

\[ = U_{k(2n+4)} - U_{k(2n+1)} + (V_k + V_{2k}) U_{k(n+2)} + U_{3k}. \]

By (13) and (14), for even number \( n \), the desired results are obtained. Similarly, for \( n = 1,3(\text{mod } 4) \), the remaining results are proven. The proof of the other result is hold. Thus, the proof is completed.

**Theorem 2.4.** Let \( k \) be an odd integer number. We have

\[ \Delta U_{2k} \sum_{i=0}^{n} (-1)^{(i+2)} a_{k,n-i}(i) \]

\[ = \begin{cases} 
V_k \left( U_{k(n+1)} + U_{kn} \right) (2 - V_{kn}), & \text{if } n \equiv 0(\text{mod } 4) \\
\Delta U_k \left( U_k^2 - U_{k(n+1)} U_{kn} - U_{kn}^2 \right), & \text{if } n \equiv 1(\text{mod } 4) \\
-\Delta U_k U_k U_{k(n+1)} U_{kn}, & \text{if } n \equiv 2(\text{mod } 4) \\
-\Delta U_k U_{kn} (V_k - 2), & \text{if } n \equiv 3(\text{mod } 4)
\end{cases} \]
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and

\[ \Delta V_k V_{2k} U_{3k} \sum_{n=0}^{n} (-1)^{n+1} a_{k,n} (i) \]

\[ V_{2k} \left( U_{3k} - U_{2(3n+4)} - U_{2(3n+1)} \right) + U_{2k} \left( V_{k(2n+3)} + \Delta U_{k} U_{3k} \right) + V_{k}^2 U_{k(n+1)} + V_{k}^2 U_{k(n+2)} + V_{k}^2 U_{k(n+3)} + U_{3k} (V_k - 1) \]

\[ = \Delta U_{2k} V_{2k} \left( -U_{2(3n+4)} - U_{2(3n+1)} \right) + (V_k - V_{2k}) U_{k(n+2)} + U_{3k} (V_k - 1) + V_{k} U_{2k} U_{k(n-1)} + 2U_{3k} V_{4k} + U_{4} V_{k(n-1)} \]

if \( n \equiv 0 \pmod{4} \)

if \( n \equiv 1 \pmod{4} \)

if \( n \equiv 2 \pmod{4} \)

if \( n \equiv 3 \pmod{4} \)

Proof. Let \( n \equiv 0 \pmod{4} \). Consider that

\[ \sum_{i=0}^{n} (-1)^{i+1} a_{k,n} (i) = -a_{k,n} (0) - a_{k,n} (1) + a_{k,n} (2) + \ldots + a_{k,n} (n-1) - a_{k,n} (n) \]

\[ = -\sum_{i=1}^{n/4} a_{k,n} (4i - 4) \sum_{i=1}^{n/4} a_{k,n} (4i - 3) + \sum_{i=1}^{n/4} a_{k,n} (4i - 2) \sum_{i=1}^{n/4} a_{k,n} (4i - 1) - a_{k,n} (n) \]

From (7) and Lemma 2.1, we write

\[ \sum_{i=0}^{n} (-1)^{i+1} a_{k,n} (i) = -\frac{U_{kn}^2}{V_k} \left( U_{k(n+1)} + U_{kn} \right) \sum_{i=1}^{n/4} \left( U_{k(n-4i+3)} - U_{k(4i-3)} \right) \]

By (5), (7) and (8), we have

\[ \sum_{i=0}^{n} (-1)^{i+1} a_{k,n} (i) = -\frac{1}{V_k} U_{kn}^2 - \frac{V_k}{\Delta U_{2k}} \left( U_{k(n+1)} + U_{kn} \right) (V_{kn} - 2) \]

if \( n \equiv 0 \pmod{4} \)

if \( n \equiv 1 \pmod{4} \)

as claimed. For \( n \equiv 1, 2, 3 \pmod{4} \), the proofs are clearly given. Similarly, the other result is given. Thus, we have the conclusion.

REFERENCES (KAYNAKÇA)


